13.4 Triple Integrals (Definite)

$$\lim_{\substack{\Delta V \to 0 \\ N \to \infty}} \sum_{k=1}^{N} f(x_k, y_k, z_k)\,\Delta V = \iiint_D f(x,y,z)\,dx\,dy\,dz$$

For cartesian parametrization $\Delta V = dx\,dy\,dz$
But for cylindrical polar or spherical ΔV has to be different, as we shall see.

Take a region D in 3D space:

Let $D = \{(x,y,z) : a \le x \le b,\ g(x) \le y \le h(x),\ G(x,y) \le z \le H(x,y)\}$

(Note: D as parametrized above is not unique)

and some function $f = f(x,y,z)$

$$\iiint_D f\,dV = \int_a^b dx \left\{ \int_{g(x)}^{h(x)} dy \left[\int_{G(x,y)}^{H(x,y)} dz\ f(x,y,z) \right] \right\}$$

First, integrate [ ] in z: you obtain a function of x, y.
Next, integrate { } in y: you obtain a function of x.

Finally, integrate in x: you obtain a number.

Why this order? Because of D parametrization.

S'pose $D = \{(x,y,z) : c \leq z \leq d, \alpha(z) \leq x \leq \beta(z), A(x,z) \leq y \leq B(x,z)\}$

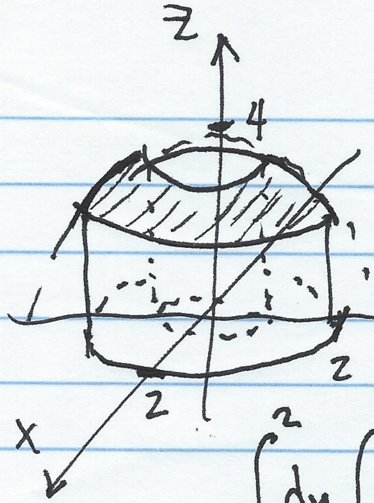
then

$$\iiint_D f\,dV = \int_c^d dz \int_{\alpha(z)}^{\beta(z)} dx \int_{A(x,z)}^{B(x,z)} dy$$

ex) We'll do a problem with cylindrical symmetry using a cartesian parametrization. In the next section we'll do the problem using cylindrical polar coordinates which exploit the symmetries to make the calculation easier.

EXAMPLE (7)

ex) Find volume of solid, aligned with z-axis. The region consists of a cylinder of radius=2 (outer) and radius=1 (inner), bounded by $z \geq 0$ and $z = 4 - x^2 - y^2$



We will find the volume of a solid with radius $r=2$ & subtract the volume of the solid with radius $r=1$

$$\int_{-2}^{2} dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx \int_{0}^{4-x^2-y^2} dz \;-\; \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \int_{0}^{4-x^2-y^2} dz$$

$$= \int_{-2}^{2} dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx \,(4-x^2-y^2) - \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \,(4-x^2-y^2) \qquad (\$)$$

They are similar integrals, so let's solve

$$\int_{-a}^{a} dy \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dx \,(4-x^2-y^2)$$ and then replace a by 2 and 1

$$= \int_{-a}^{a} dy \left[4x - \frac{1}{3}x^3 - xy^2 \right|_{x=-\sqrt{a^2-y^2}}^{x=\sqrt{a^2-y^2}}$$

$$= \int_{-a}^{a} dy \left[8\sqrt{a^2-y^2} - \frac{2}{3}(a^2-y^2)^{3/2} - 2y^2(a^2-y^2)^{1/2} \right]$$

we apply a trig substitution: let $y = a\sin\theta$, $dy = a\cos\theta\, d\theta$

$$\begin{cases} y = a \Rightarrow \theta = \pi/2 \\ y = -a \Rightarrow \theta = -\pi/2 \end{cases}$$

$$\int_{-\pi/2}^{\pi/2} a\cos\theta\, d\theta \left[8\sqrt{a^2-a^2\sin^2\theta} - \frac{2}{3}\left(a^2-a^2\sin^2\theta\right)^{3/2} - 2a^2\sin^2\theta\left(a^2-a^2\sin^2\theta\right)^{1/2}\right]$$

$$= \int_{-\pi/2}^{\pi/2} a\cos\theta\, d\theta \left[a8\cos\theta - \frac{2}{3}a^3\cos^3\theta - 2a^3\sin^2\theta\cos\theta\right]$$

$$= 2a^2\int_{-\pi/2}^{\pi/2} \left(4 - a^2\sin^2\theta\right)\cos^2\theta\, d\theta - \frac{2a^4}{3}\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta$$

use the following identities

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

to obtain:

$$2a^2\left\{\int_{-\pi/2}^{\pi/2} d\theta\left[2 - \frac{3}{8}a^2 + \cancelto{0}{2\cos 2\theta} - \cancelto{0}{\frac{a^2}{8}\cos 4\theta}\right]\right\} - \frac{2a^4}{3}\left\{\int_{-\pi/2}^{\pi/2} d\theta\left[\frac{3}{8} + \cancelto{0}{\frac{1}{2}\cos 2\theta} + \cancelto{0}{\frac{1}{8}\cos 4\theta}\right]\right\}$$

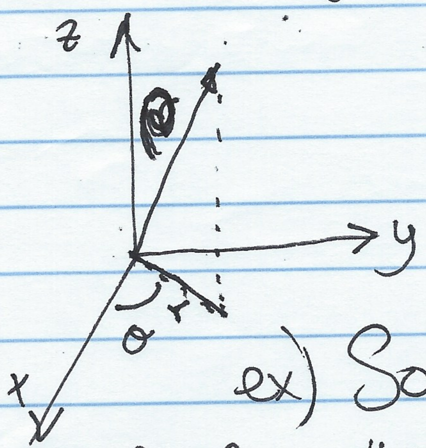
$\nearrow 0$ are zero because they are integrals of periodic functions over $\theta = [-\frac{\pi}{2}, \pi/2]$

$$\therefore \int = 2a^2\pi\left[2 - \frac{1}{2}a^2\right]$$

Going back to ($) using (#) with $a = 2$ & 1

$$\$ = \int_{a=2} - \int_{a=1} = 2\cdot 4\pi\left[2-2\right] - 2\pi\left[2-\tfrac{1}{2}\right] = 3\pi$$

Triple Integral using cylindrical/polar



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \\ r^2 = x^2 + y^2 \end{cases}$$

$dV = dz\, r\, d\theta\, dr$

$\rho = \sqrt{r^2 + z^2}$ $\qquad \tan\theta = \frac{y}{x}$

ex) Solve (#) EXAMPLE using cylindrical

$$\int_0^{2\pi} d\theta \int_1^2 r\,dr \int_0^{4-r^2} dz = 2\pi \int_1^2 r\,dr\,(4-r^2)$$

let $u = 4 - r^2$ $\quad du = -2r\,dr$

$r = 1 \quad u = 3 \qquad r = 2 \quad u = 0$

$$-\frac{2\pi}{2}\int_3^0 u\,du = \pi \int_0^3 u\,du = \frac{\pi}{2}u^2\Big|_0^3 = \frac{9\pi}{2}$$

A lot easier!!

ex) Volume of hemisphere $z>0$ $x^2+y^2+z^2 \leq a^2$

Cartesian: using only 1 quadrant and then multiplying by 4:

$$V = 4\int_0^a dx \int_0^{(a^2-x^2)^{1/2}} dy \int_0^{(a^2-x^2-y^2)^{1/2}} dz$$

$$= 4\int_0^a dx \int_0^{(a^2-x^2)^{1/2}} dy \, (a^2-x^2-y^2)^{1/2} \quad \text{(not an easy integral!)}$$

$$= 4\int_0^a dx \, \frac{1}{2}\left[y(a^2-x^2-y^2)^{1/2} + (a^2-x^2)\arcsin\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]\Bigg|_{y=0}^{y=(a^2-x^2)^{1/2}}$$

$$= 4\int_0^a \left[\left(\frac{a^2-x^2}{2}\right)\frac{\pi}{2}\right] dx = \frac{4\pi}{4}\,\frac{2a^3}{3} = \frac{2\pi a^3}{3} \quad //$$

Using cylindrical/polar:

$$V = \int_0^a r\,dr \int_0^{2\pi} d\theta \int_0^{(a^2-r^2)^{1/2}} dz = 2\pi \int_0^a r\,dr\,(a^2-r^2)^{1/2} = -\pi \int_{a^2}^0 u^{1/2}\,du$$

$$u = a^2 - r^2 \quad du = -2r\,dr$$

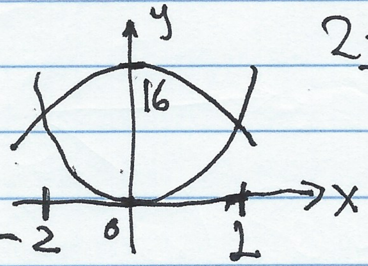
$$= \pi \int_0^{a^2} u^{1/2}\,du = \frac{\pi 2}{3} a^3 \quad //$$

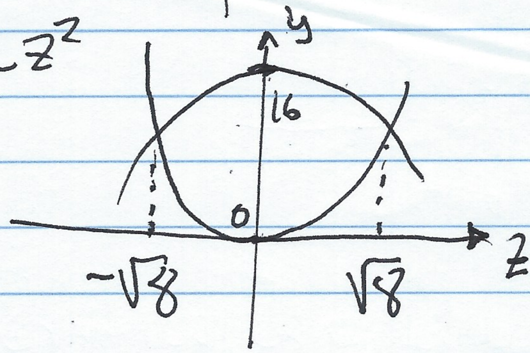
ex) Here's another example. We quickly get to a simple-looking integral using cylindrical coordinates, however, to solve the integral we need to use "complex variables" (a more advanced type of calculus)

Find volume of region D bounded by paraboloids

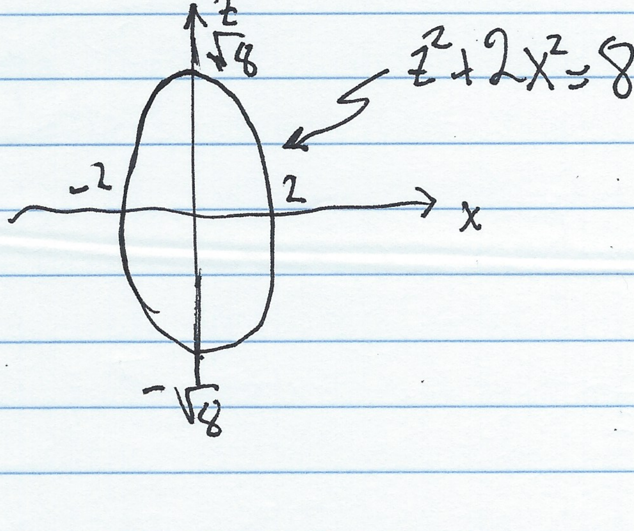
$y = x^2 + z^2$ $\quad$ $y = 16 - 3x^2 - z^2$

2 views:





The shadow of region on x,z plane



In cartesian

$$V = \int_{-2}^{2} dx \int_{-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} dz \int_{x^2+z^2}^{16-3x^2-z^2} dy$$

$$= \int_{-2}^{2} dx \int_{-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} dz \left(16-3x^2-z^2-x^2-z^2\right) = \int_{-2}^{2} dx \int_{-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} dz \left(16-4x^2-2z^2\right)$$

$$= \int_{-2}^{2} dx \left[16z - 4x^2 z - \frac{2}{3}z^3\right]_{z=-\sqrt{8-2x^2}}^{z=\sqrt{8-x^2}}$$

after a lot of algebra

$$= \int_{-2}^{2} dx\, \frac{16\sqrt{2}}{3}\left(4-x^2\right)^{3/2}$$

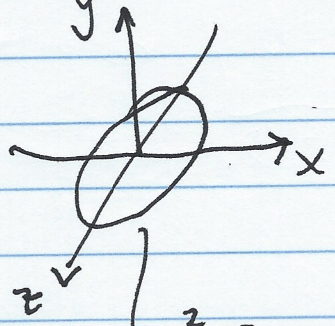
let $x = 2\sin\theta$
$dx = 2\cos\theta\, d\theta$

$$= \frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} 2^4 \cos^4\theta\, d\theta = \frac{16^2\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \frac{1}{4}\left(1+\cos 2\theta\right)^2 d\theta$$

$$= \frac{16^2}{3\cdot 4}\sqrt{2}\int_{-\pi/2}^{\pi/2}\left(\frac{3}{2} + \cancel{2\cos 2\theta}^{\,0} + \cancel{\frac{1}{2}\cos 4\theta}^{\,0}\right) d\theta = \frac{16^2}{\cancel{3}\cdot 4}\sqrt{2}\,\frac{\cancel{3}}{2}\pi = 32\sqrt{2}\pi$$

$32\sqrt{2}\pi$

Now, using cylindrical coordinates:



Shadow of region

$$\left.\begin{aligned} y_1 &= 16-3x^2-z^2 \\ y_2 &= x^2+z^2 \end{aligned}\right\} y_1-y_2 = 16-2z^2-4x^2$$

$$z^2+2x^2=8 = \underbrace{\underbrace{z^2+x^2}_{r^2}+x^2=8}$$

$x = r\cos\theta$ $\quad x^2+z^2=r^2$

$z = r\sin\theta$

$$r^2 + r^2\cos^2\theta = 8$$

$$r^2(1+\cos^2\theta) = 8$$

$$\therefore r = \frac{\sqrt{8}}{(1+\cos^2\theta)^{1/2}} = \frac{\sqrt{8}}{f(\theta)} \qquad f(\theta) \equiv (1+\cos^2\theta)^{1/2}$$

$$V = \int_0^{2\pi} d\theta \int_0^{\sqrt{8}/f(\theta)} r\,dr \int_{y_2}^{y_1} dy\, \cancel{(y_1-y_2)} = \int_0^{2\pi} d\theta \int^{\sqrt{8}/f(\theta)} r\,dr\,(y_1-y_2)$$

$$y_1 - y_2 = 16 - 2r^2\cos^2\theta - 2r^2$$

$$V = \int_0^{2\pi} d\theta \int_0^{\sqrt{8}/f(\theta)} dr\,\Big[16r - 2r^3\,\overbrace{[\cos^2\theta+1]}^{f^2(\theta)}\Big]$$

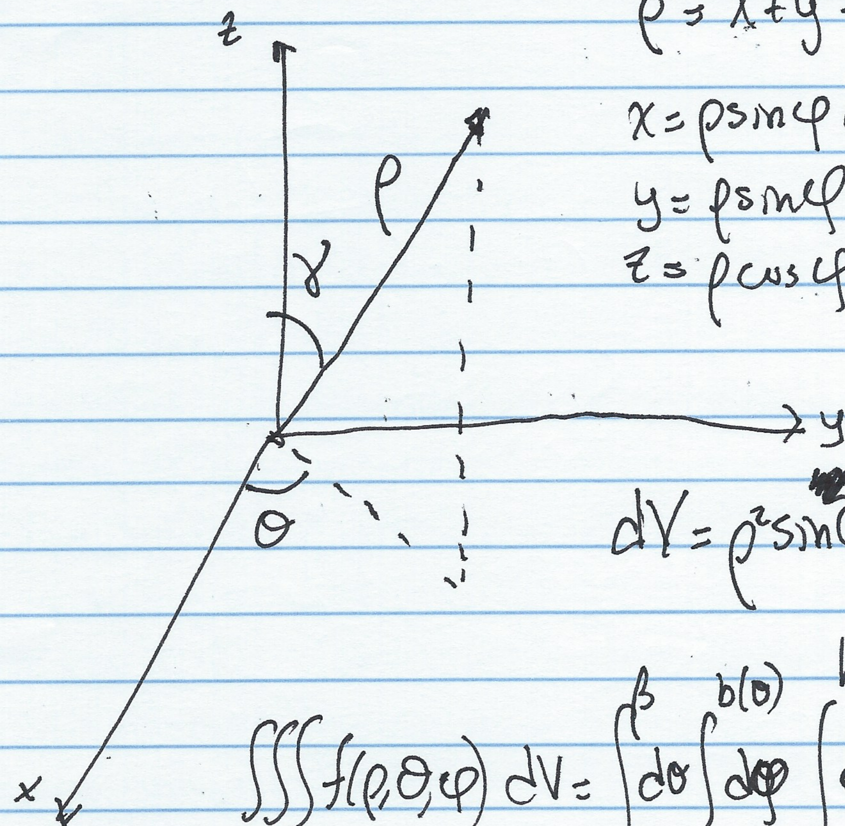
$$= \int_0^{2\pi} d\theta\,\left(8r^2 - \frac{1}{2}r^4 f^2(\theta)\right)\Bigg|_{r=0}^{r=\sqrt{8}/f(\theta)}$$

$$\int_0^{2\pi} d\theta \left[\frac{8^2}{f^2(\theta)} - \frac{8^2}{2}\frac{f^2(\theta)}{f^4(\theta)} \right] = \int_0^{2\pi} d\theta \frac{1}{f^2(\theta)} \left[8^2 - \frac{8^2}{2} \right]$$

$$= \frac{8^2}{2} \int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta} = 32 \underbrace{\int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta}}_{} = 32\sqrt{2}\,\pi$$

this integral can be done using complex analysis. //

TRIPLE INTEGRALS USING SPHERICAL COORDINATES



$$\rho^2 = x^2 + y^2 + z^2$$

$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\varphi$$

$$z = \rho \cos\varphi$$

$$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\iiint_R f(\rho,\theta,\varphi)\, dV = \int_\alpha^\beta d\theta \int_{a(\theta)}^{b(\theta)} d\varphi \int_{g(\theta,\varphi)}^{h(\theta,\varphi)} d\rho\, f(\rho,\theta,\varphi)\, \rho^2 \sin\varphi$$

ex) $V = \frac{4\pi R^3}{3}$ is the volume of a sphere of radius R

let's see if we recover this result

$$V = \int_0^{\pi} d\varphi \int_0^{2\pi} d\theta \int_0^{R} d\rho\ \rho^2 \sin\varphi = 2\pi \int_0^{\pi} d\varphi \sin\varphi \int_0^{R} \rho^2 d\rho$$

$$\frac{2\pi R^3}{3} \int_0^{\pi} d\varphi \sin\varphi = \frac{2\pi R^3}{3} \left(-\cos\varphi\right|_0^{\pi} = \frac{4\pi R^3}{3}$$

ex) Calculate

$$\cancel{\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}} \quad \iiint_D \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$$

D is a region between the spheres of radius 1 & 2

$$\int_0^{\pi} d\varphi \int_0^{2\pi} d\theta \int_1^{1} d\rho\, \rho^2 \sin\varphi \frac{1}{\rho^3} = 2\pi \int_0^{\pi} d\varphi \sin\varphi \int_1^{2} \frac{d\rho}{\rho}$$

$$= -2\pi \cos\varphi\Big|_0^{\pi} \ln\rho\Big|_1^{2} = -2\pi(\cos\pi - \cos 0)\ln 2$$

$$= 4\pi \ln 2$$