13.3 Double & Triple Integrals

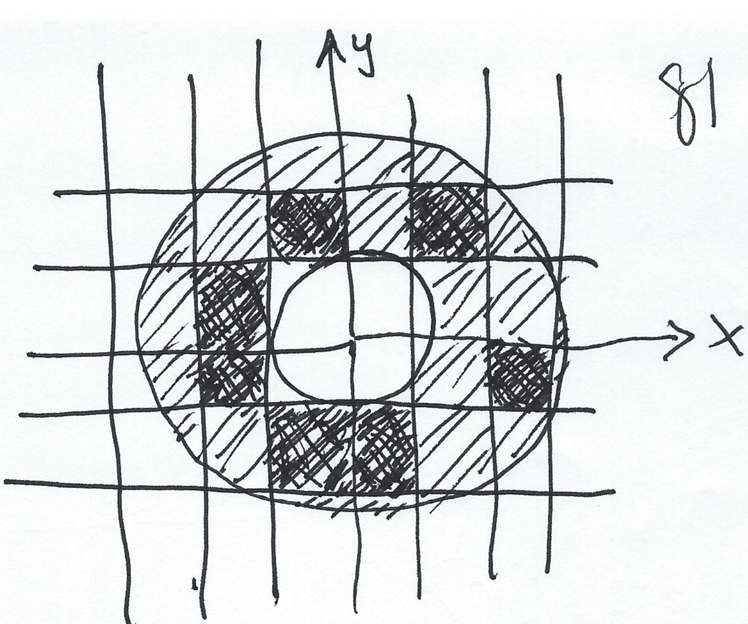
(with polar/cylindrical & spherical symmetries)

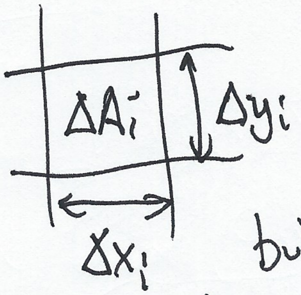
The Big Picture: we want to find ways to integrate, or that make it easier to integrate; when symmetries arise we exploit these to form-fit differential elements which are then easy to "add".

Here's the intuition: suppose we want to find the area of a disk.

$$A \approx \sum_{i=1}^{N} \Delta A_i$$

where ΔA_i are <u>rectangular</u> elements of area $\Delta x_i \Delta y_i$





The darker elements are rectangular but the others shaded areas are poorly approximated by rectangles. If smaller rectangles are chosen, (N increases) there would be a larger density of "fitting" elements. The elements that won't form fit would have a smaller percentage of their area outside of the disk.

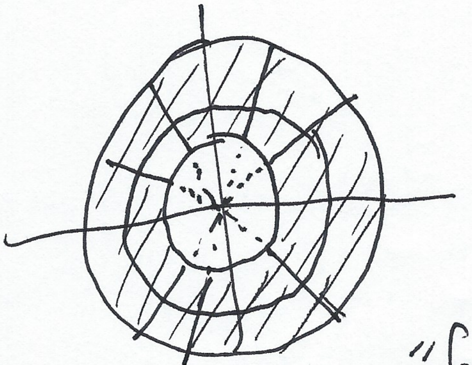
The Riemann sum is based upon the above sum. However the process is designed so that the area of each element gets smaller at the <u>same rate</u> as the number of ΔA_i increases. In the limit

$$A = \int_{\text{Disk}} dA = \iint_{\text{Disk}} dx\, dy$$

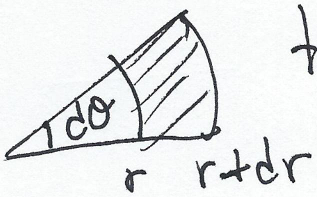
The important point here is that the area can be calculated by using a logically-rectangular grid. Always.

<u>However</u> using a cartesian grid to parametrize something with cylindrical symmetry will lead to a coupling of the integration variables x, y via the parametrization of the boundaries.

S'pse instead we use "wedge" elements ΔA_i.

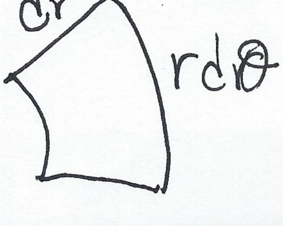


At a coarse level (large ΔA_i), things "fit" better. Here again we can choose to make the area elements small as we increase the number of elements. <u>However</u> to get the sum of wedges



to decrease in size & the number of elements to get larger to <u>balance</u> we need to multiply $dr\,d\theta$ by r. That is

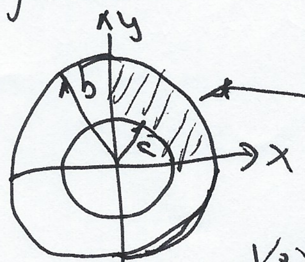
$$dA = r\,d\theta\,dr$$



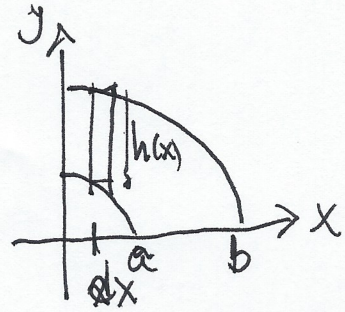
So $A = \iint_{\text{Disk}} r\,dr\,d\theta$

but what makes this integral easier to perform ~~...~~ is that the boundaries of the disk are now easy to describe in cylindrical coordinates.

ex) Find area of Disk of inner radius a & outer radius b:



cartesian: $A = 4\iint_{\text{1 quadrant}} dA$



$\cancel{dA = h(x)\,dx}$

$$A = 4\int_{0}^{b} dx \int_{y_{lower}(x)}^{y_{upper}(x)} dy$$

$y_{upper}(x) = \sqrt{b^2 - x^2}$

$$y_{lower}(x) = \begin{cases} \sqrt{a^2 - x^2} & 0 \le x \le a \\ 0 & a \le x \le b \end{cases}$$

$$A = 4\int_{a}^{b} dx \left[\sqrt{b^2 - x^2} - 0\right] + 4\int_{0}^{a} dx \left[\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right]$$

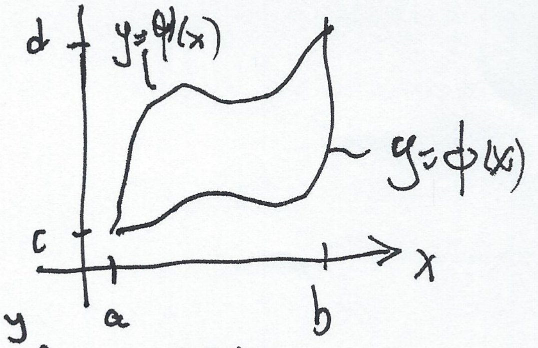
<u>Using polar</u>: $A = \int_0^{2\pi} d\theta \int_a^b r\,dr = 2\pi \int_a^b r\,dr = 2\pi \frac{1}{2} r^2 \Big|_a^b = \pi(b^2 - a^2)$

Much easier!!

The Double Integral (Definite integrals)

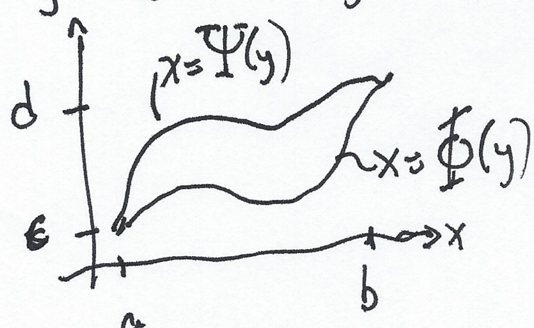
<u>Cartesian</u>

$$\int_a^b dx \int_{y=\phi(x)}^{y=\psi(x)} dy\, f(x,y)$$



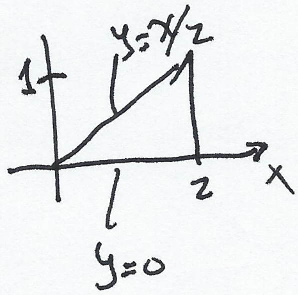
OR

$$\int_c^d dy \int_{x=\Phi(y)}^{x=\Psi(y)} dx\, f(x,y)$$



They give same answer, the choice is dictated by how easy it is to perform the integral

ex) Find $\cancel{\iint_D xy^2\, dA}$ $\iint_D xy^2\, dx\,dy$

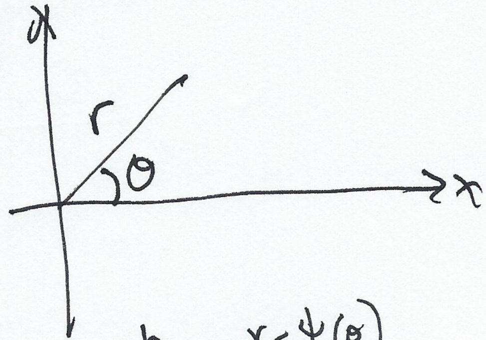
Where D: 

D: $0 \le x \le 2,\ 0 \le y \le x/2$

$$\int_0^2 dx \left\{ \int_0^{x/2} xy^2\, dy \right\} = \int_0^2 dx \left(\frac{xy^3}{3} \Big|_{y=0}^{y=x/2} \right)$$

$$= \int_0^2 dx \left(\frac{x}{3}\left(\frac{x}{2}\right)^3 - 0 \right) = \int_0^2 dx\, \frac{x^4}{24} = \frac{x^5}{5\cdot 24}\Big|_0^2 = \frac{32}{5\cdot 24} = \frac{4}{15}$$ //

(Definite) Double Integral (Polar Coordinates)



$x = r\cos\theta$

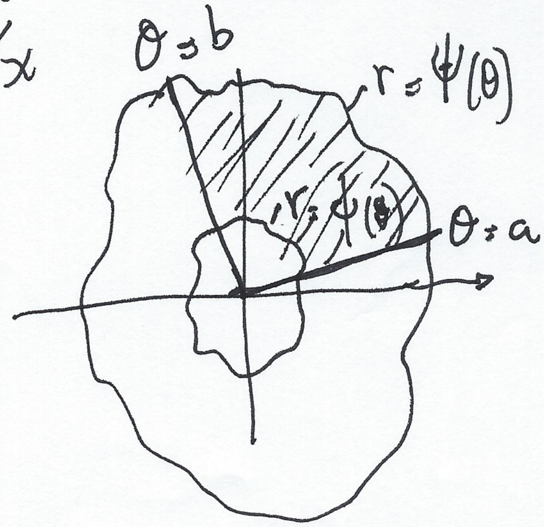
$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$\tan\theta = y/x$

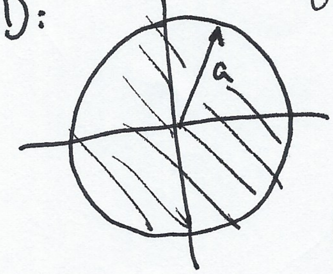
$$\boxed{dA = r\,dr\,d\theta}$$

$$\int_a^b d\theta \int_{r=\phi(\theta)}^{r=\psi(\theta)} r\,dr\, f(r,\theta)$$



ex) find $\int_D f(x,y)\,dA$ using polar:

D:



$0 \le r \le a\,,\ 0 \le \theta \le 2\pi$

$$f = \sqrt{a^2-x^2-y^2} = \sqrt{a^2-r^2}$$

$$\int f(x,y)\,dA = \int_0^{2\pi} d\theta \int_0^a r\,dr\sqrt{a^2-r^2} = 2\pi\int_0^a dr\, r\sqrt{a^2-r^2}$$

let $u = a^2 - r^2$ $\quad \begin{cases} r=0 & u=a^2 \\ r=a & u=0 \end{cases}$

$du = -2r\,dr$

$$= -\frac{2\pi}{2}\int_{a^2}^{0} u^{1/2}\,du = \pi\int_0^{a^2} u^{1/2}\,du = \frac{2\pi}{3}u^3\Big|_0^{a^2} = \frac{2\pi a^6}{3}$$

//

Ex) Here's an example where adopting polar coordinates is the difference between being able or not to perform the integral:

$$\text{find } \iint_{\text{Disk}} \left[e^{-\frac{1}{8}(x^2+y^2)} - e^{-2}\right] dx\,dy$$

Disk is of radius 4.

Here, switching from integrating first in x or y makes no difference. We're stuck!

Polar: $r^2 = x^2+y^2$ Disk: $\{0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$

$$e^{-\frac{1}{8}(x^2+y^2)} - e^{-2} = e^{-\frac{1}{8}r^2} - e^{-2}$$

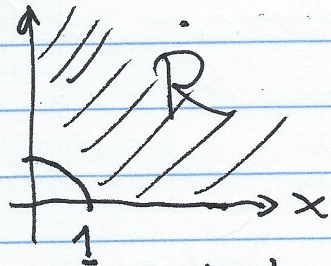
$$dx\,dy \rightarrow r\,dr\,d\theta$$

$$\int_0^{2\pi} d\theta \int_0^4 r\,dr\left[e^{-\frac{1}{8}r^2} - e^{-2}\right] = 2\pi \int_0^4 dr\, r\left[e^{-\frac{1}{8}r^2} - e^{-2}\right] = 2\pi\left\{\int_0^4 dr\, r e^{-\frac{1}{8}r^2} - e^{-2}\int_0^4 r\,dr\right\} = 2\pi\left\{-4\int_0^{-2} e^u\,du - e^{-2}\tfrac{1}{2}r^2\Big|_0^4\right\} = 2\pi\left\{-4e^u\Big|_0^{-2} - \frac{e^{-2}}{2}(16)\right\}$$

$$= 2\pi\left(-4e^{-2} + 4 - 8e^{-2}\right) = 2\pi\left(4 - 12e^{-2}\right) = 8\pi\left(1 - 3e^{-2}\right)$$

Another good example of the utility of the polar transformation. You want to evaluate the improper integral

$$\iint_R \frac{x}{(x^2+y^2)^2}\,dx\,dy \qquad R=\left\{(r,\theta) : 1\le r<\infty;\ 0\le\theta\le\frac{\pi}{2}\right\}$$



$x = r\cos\theta$
$y = r\sin\theta$

$$\int_0^{\pi/2} d\theta \int_1^{\infty} r\,dr\,\frac{r\cos\theta}{\underbrace{(r^2\cos^2\theta + r^2\sin^2\theta)^2}_{= r^4}} = \int_1^{\infty}\frac{dr}{r^2}\int_0^{\pi/2}\cos\theta\,d\theta = \int_1^{\infty}\frac{dr}{r^2}\sin\theta\Big|_0^{\pi/2}$$

$$= \int_1^{\infty}\frac{dr}{r^2} = \lim_{p\to\infty}\int_1^{p}\frac{dr}{r^2} = \lim_{p\to\infty} -r^{-1}\Big|_1^{p} = \lim_{p\to\infty}\left(-\frac{1}{p}+1\right) = 1$$