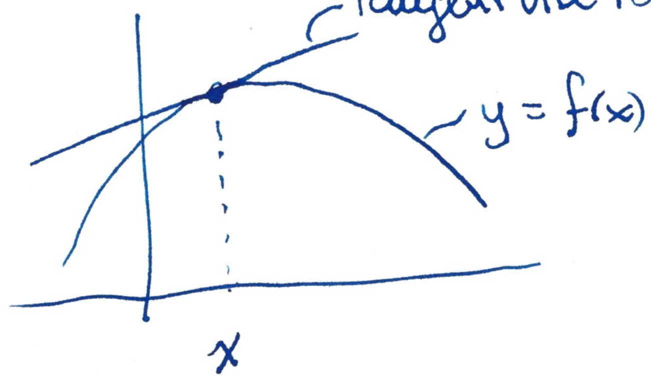
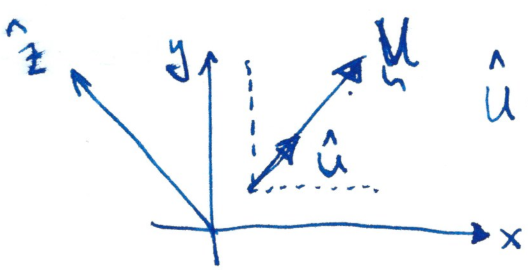
12.6 Directional Derivative & the Gradient Operator

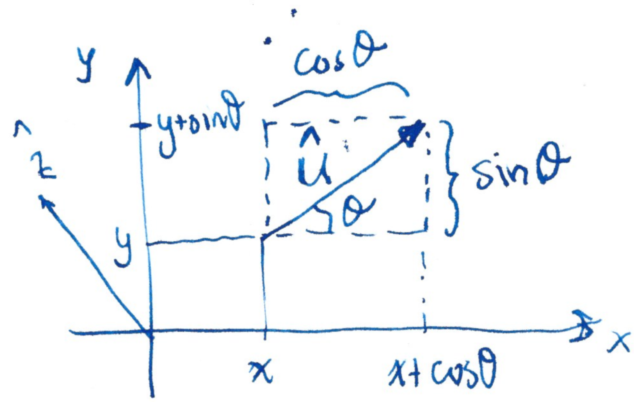
$\frac{df(x)}{dx}$: The derivative of $f(x)$ $\equiv$ rate of change wrt x of $f(x)$



in 3D: We can ask for the rate of change in a certain direction $\hat{u}$



$\hat{u} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$



$|\hat{u}| = 1$

~~$\hat{u}_1 = \langle \sin\theta, \cos\theta, 0 \rangle$~~

$\hat{u} = \langle \cos\theta, \sin\theta \rangle$

The Directional Derivative is the derivative of $z = f(x,y)$ in the direction $\hat{u}$. It is

$$D_{\hat{u}} f(x,y) = \lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h}$$

Take $|h| \ll 1$ then

$$f(x + h\cos\theta, y + h\sin\theta) \approx f(x,y) + h\cos\theta \frac{\partial f}{\partial x} + h\sin\theta \frac{\partial f}{\partial y}$$

$$\therefore \frac{1}{h}\left[f(x + h\cos\theta, y + h\sin\theta) - f(x,y)\right] \approx \cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}$$

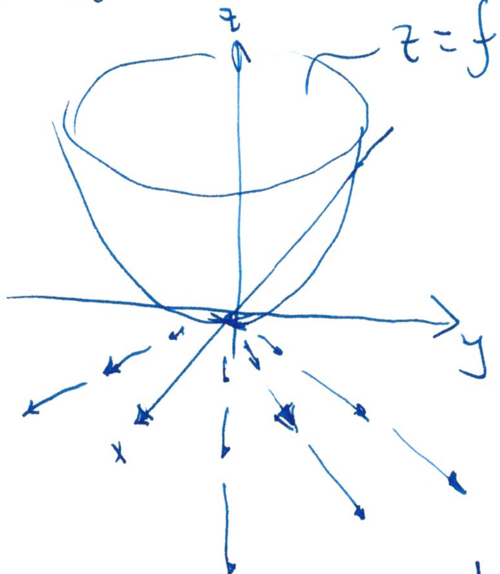
The Gradient $\nabla f(x,y) \equiv \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ (2D Gradient)

(Takes a scalar & returns a vector)

if $\hat{u} = \langle \cos\theta, \sin\theta \rangle$

$$\therefore \boxed{\hat{u} \cdot \nabla f = \cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y} = D_{\hat{u}} f(x,y)}$$

S'pse we ask "What is the direction $\hat{u}$ of MAXIMAL ascent?"



choose θ s.t.

$$\hat{u} \parallel \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \nabla f$$

parallel

so $\hat{u} = \dfrac{1}{|\nabla f|} \nabla f$ maximal ascent

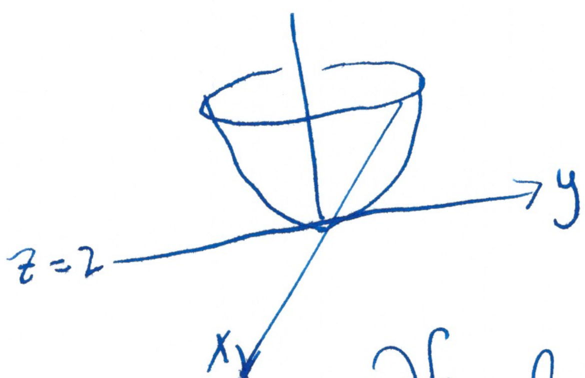
$\hat{u} = -\dfrac{1}{|\nabla f|} \nabla f$ maximal descent

Why Maximal? if $\hat{u}$ is pointing in direction ∇f

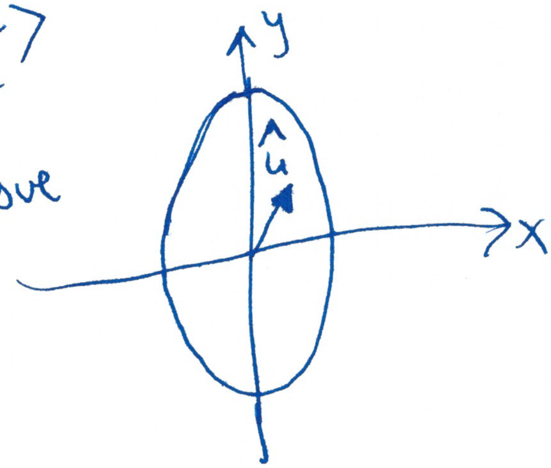
$$D_{\hat{u}} f = \frac{1}{|\nabla f|} \nabla f \cdot \nabla f = |\nabla f| \quad \text{(largest)}$$

ex) Find $D_{\hat{u}}f(1,2)$ of $z = x^2+2y^2+2 = f(x,y)$

in direction $\hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$



from above



$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 4y$$

$$D_{\hat{u}}f = \hat{u} \cdot \nabla f = \frac{1}{\sqrt{2}}2x + \frac{1}{\sqrt{2}}4y$$

$$D_{\hat{u}}f(1,2) = \sqrt{2} + \frac{8}{\sqrt{2}} = \frac{1}{\sqrt{2}}10$$

At $x=1$, $y=2$ find direction of <u>steepest descent</u>

would point in $-\nabla f$ direction

$$-\nabla f = \langle -2x, -4y \rangle$$

at $x=1, y=2$ $\quad -\nabla f(1,2) = -\langle 2,8 \rangle$

The Gradient in 3D is the operator $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

ex) Find $\nabla f(r)$

where $r = \sqrt{x^2+y^2+z^2}$

(A) $\nabla f = \hat{i}\frac{\partial f}{\partial x}(r) + \hat{j}\frac{\partial f(r)}{\partial y} + \hat{k}\frac{\partial f(r)}{\partial z}$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x}$ $\qquad$ $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$

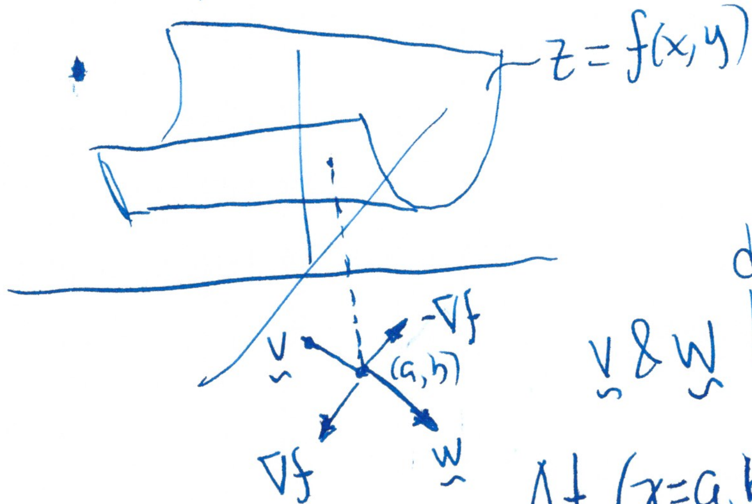
$\frac{\partial f}{\partial y}$ similar to $\frac{\partial f}{\partial x}$ $\qquad$ $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

substitute in (A):

$\nabla f = (\hat{i}x + \hat{j}y + \hat{k}z)\frac{1}{r}\frac{\partial f}{\partial r}$ $\qquad$ but if $\underset{\sim}{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$\nabla f = \hat{r}\frac{\partial f}{\partial r}$ $\qquad$ where $\hat{r} = \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|}$

let $z=f(x,y)$. If ∇f at some $(x,y)=(a,b)$ points in direction of steepest ascent & $-\nabla f$ points in direction of steepest descent <u>and</u> $\nabla f(x,y)$ is continuous, there's got to be a direction(s) of zero ascent/descent.

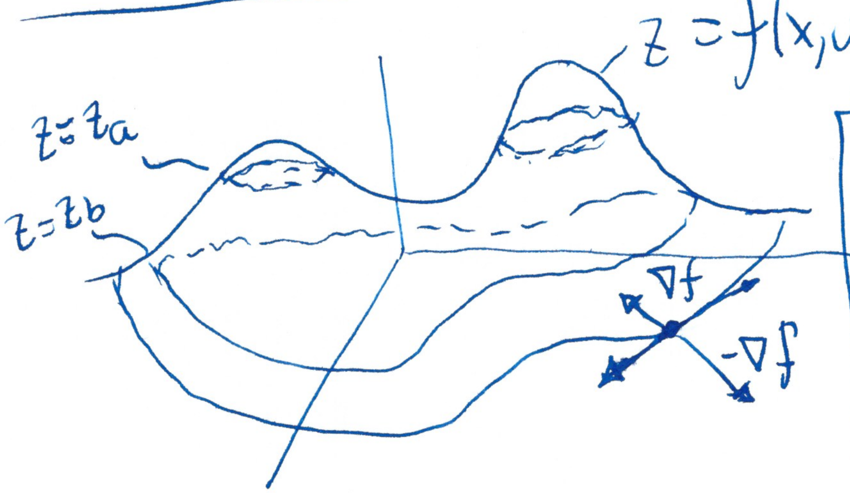


directions that $\underset{\sim}{v}$ & $\underset{\sim}{w}$ have no ascent/descent:

At $(x=a, y=b)$ these are the directions in which $f(x,y)$ does not change!

We can use this to find the "level sets" of $z=f(x,y)$

Level sets/curves (think of heights on a topo map)

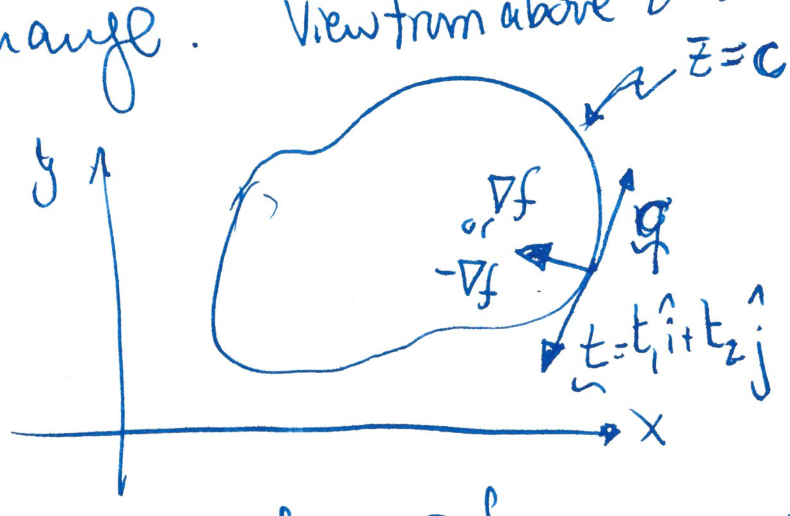


A level set consists of all $\{x, y\}$ such that

$$z = \underset{\uparrow \text{ a constant}}{C} = f(x,y)$$

How do we find level sets? Recall that there's a set of directions in which $f(x,y)$ does not change.

View from above $z=f(x,y)$

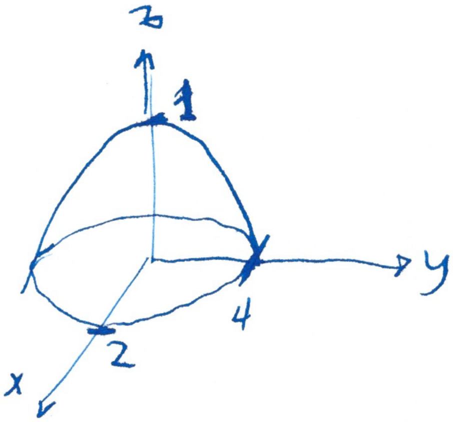


$$\underset{\uparrow \text{ 2D gradient}}{\nabla f} \cdot \underset{\sim}{t} = 0$$

$$\nabla f \cdot \underset{\sim}{t} = \frac{\partial f}{\partial x} t_1 + \frac{\partial f}{\partial y} t_2 = 0 \quad \text{let } t_1 = -\frac{\partial f}{\partial y} \Rightarrow t_2 = \frac{\partial f}{\partial x}$$

$$\therefore \underset{\sim}{t} = -\frac{\partial f}{\partial y}\hat{i} + \frac{\partial f}{\partial x}\hat{j}$$

ex) $z = f(x,y) = \frac{1}{4}\sqrt{16-4x^2-y^2}$



a) Find directions of steepest descent

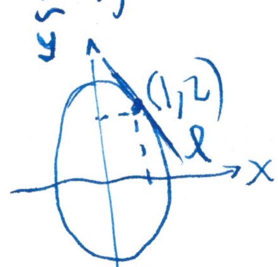
b) Find $D_{\hat{u}}f$ if $\hat{u}$ points in $\underset{\sim}{v} = 3\hat{i}$

c) Compute slope of line tangent to level set at $P = (1,2)$ & verify that tangent line is $\perp$ to ∇f at P.

a) $-\nabla f = \frac{-x}{\sqrt{16-4x^2-y^2}}\hat{i} - \frac{1}{4}\frac{y\,\hat{j}}{\sqrt{16-4x^2-y^2}} = -\frac{x\hat{i}}{4f} - \frac{\hat{j}\,y}{16f}$

b) $\hat{u} = \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = \hat{i} \qquad D_{\hat{i}}f = -\frac{x}{4f}$

c) $\underset{\sim}{t} \cdot \nabla f = 0 = t_1\frac{\partial f}{\partial x} + t_2\frac{\partial f}{\partial y} \quad \therefore t_1 = -\frac{\partial f}{\partial y} = \frac{y}{16f} \quad t_2 = \frac{-x}{4f}$



$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-x}{4f}\frac{16f}{y} = -\frac{4x}{y}$ (tangent)

Tangent line ℓ at (1,2) $\quad y = mx + b \therefore \frac{dy}{dx} = m = -\frac{4x}{y}\Big|_{\substack{x=1\\y=2}} = -2$

and must pass $\cancel{f(1,2) = \frac{1}{4}\sqrt{16-4-4}}$ through (1,2):

$$-2 = -2 + b \Rightarrow b = 4$$

so tangent line $y = -2x + 4$