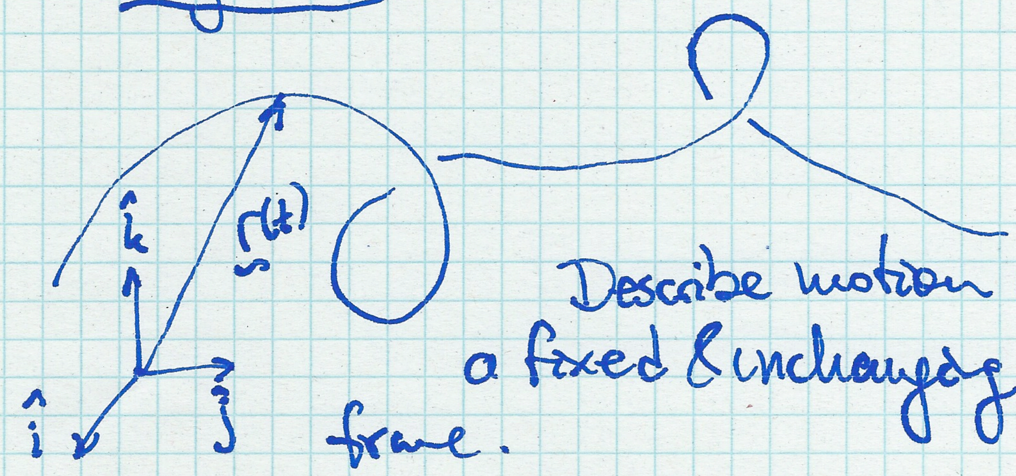
11.8 & 11.9 Non-inertial Frame or Relative Frame

Big Picture: Absolute / inertial frame



Describe motion with respect to a fixed & unchanging (in t) reference frame.

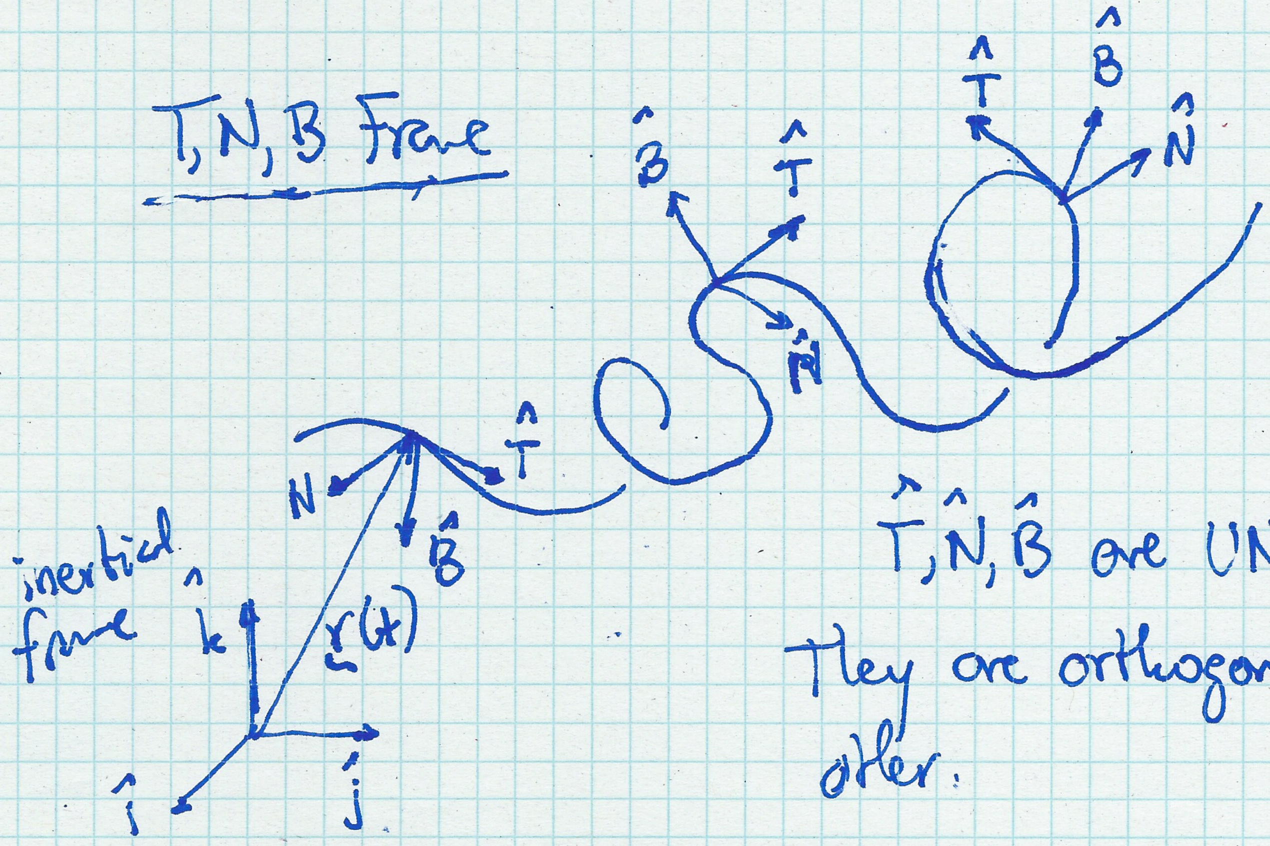
e.g. Describe motion in 3D wrt a fixed point in the universe.

In some problems more convenient to have a moving frame (often non-inertial). e.g. motion of a swarm of vehicles with respect to a central vehicle. Or a growing tree & you want to describe its mechanics in terms of basic laws of physics.

[def An inertial frame: frame in which Newton's first law is valid]

A non-inertial frame is the "T, N, B" frame which leads to the Frenet-Serret formulas

T, N, B Frame



$\hat{T}, \hat{N}, \hat{B}$ are UNIT vectors.

They are orthogonal to each other.

The tangent vector $\hat{T} \equiv \dfrac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = \dfrac{\underset{\sim}{v}}{v}$
(in the direction of $\underset{\sim}{v}(t)$)
tangent to the path.

The Normal vector $\hat{N}$ (will be defined shortly).

The Binormal vector $\hat{B} = \hat{T} \times \hat{N}$

so if you know $\hat{T}$ & $\hat{N}$ you can find $\hat{B}$.

let $s(t) = \int_{t_0}^{t} v(\xi)\, d\xi$ $\qquad v(t) = \frac{ds}{dt}$ speed

is the <u>arclength</u> of a path at time t, wrt $s(t_0)$.

$$s(t) = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

where $\underset{\sim}{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $v = |\underset{\sim}{v}|$

$$\boxed{\hat{T} = \frac{\underset{\sim}{v}(t)}{v(t)} = \frac{d\underset{\sim}{r}}{ds}}$$. To see this:

$$\frac{d\underset{\sim}{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

since $\frac{ds}{dt} = v \implies \frac{dt}{ds} = \frac{1}{v}$

$$\hat{T} = \frac{1}{v}\frac{d\underset{\sim}{r}}{dt} = \frac{1}{v}\frac{d\underset{\sim}{r}}{ds}\frac{ds}{dt} = \frac{1}{v}\frac{d\underset{\sim}{r}}{ds} v = \frac{d\underset{\sim}{r}}{ds} \quad //$$

$$\text{so} \left[\underset{\sim}{V} = \frac{ds}{dt}\hat{T} \right]$$

Look at acceleration:

$$\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\hat{T}\right) = \frac{d^2s}{dt^2}\hat{T} + \frac{ds}{dt}\frac{d\hat{T}}{dt}$$

but $\frac{d\hat{T}}{dt} = \frac{d\hat{T}}{ds}\frac{ds}{dt}$

$$\boxed{\underset{\sim}{a} = \frac{d^2s}{dt^2}\hat{T} + \left(\frac{ds}{dt}\right)^2\frac{d\hat{T}}{ds}} \quad (*)$$

To interpret, look at what $\frac{d\hat{T}}{ds}$:

Note: $\hat{T}(t)\cdot\hat{T}(t) = 1$ differentiate:

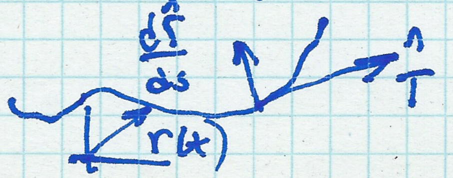
$$\frac{d\hat{T}}{dt}\cdot\hat{T} + \hat{T}\cdot\frac{d\hat{T}}{dt} = 0 \quad \text{or} \quad 2\hat{T}\cdot\frac{d\hat{T}}{dt} = 0$$

so $\hat{T}$ and $\frac{d\hat{T}}{dt}$ are perpendicular (or 1 or both 0)

by chain rule:

$$\frac{d}{dt}(\hat{T}\cdot\hat{T}) = \frac{ds}{dt}\frac{d}{ds}(\hat{T}\cdot\hat{T}) = v\frac{d}{ds}(\hat{T}\cdot\hat{T}) = 2v\,\hat{T}\cdot\frac{d\hat{T}}{ds} = 0$$

so $\hat{T}$ and $\frac{d\hat{T}}{ds}$ are perpendicular (or 1 or both zero)



So: $\hat{T}$ and $\frac{d\hat{T}}{ds}$ are orthogonal. $\hat{T}$ has unit magnitude, but $\frac{d\hat{T}}{ds}$ does <u>not</u> generally.

let $$\frac{d\hat{T}}{ds} = \left|\frac{d\hat{T}}{ds}\right| \hat{N} = k\hat{N}$$

so $\hat{N}$ is unit & orthogonal to $\hat{T}$

the $\left|\frac{d\hat{T}}{ds}\right| \equiv k$ we call "<u>curvature</u>" is the magnitude of $\frac{d\hat{T}}{ds}$.

Go back to ($\ast$):

$$\underset{\sim}{a} = \frac{d^2s}{dt^2}\hat{T} + v^2 k\hat{N} = \underbrace{\frac{dv}{dt}\hat{T}}_{\text{"tangential acceleration"}} + \underbrace{v^2 k\hat{N}}_{\text{"normal acceleration"}}$$

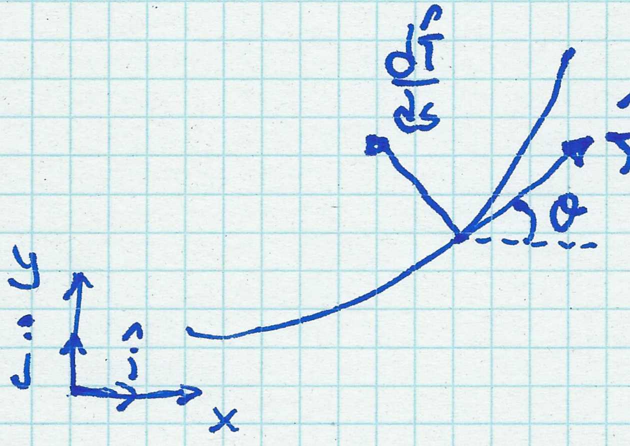
~~centripetal~~

$$\underset{\sim}{a} = a_T\hat{T} + a_N\hat{N}$$

$$a_T = \frac{dv}{dt} \qquad a_N = v^2 k$$

Look at k (the curvature)

in 2D easier to see:



$$\hat{T} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

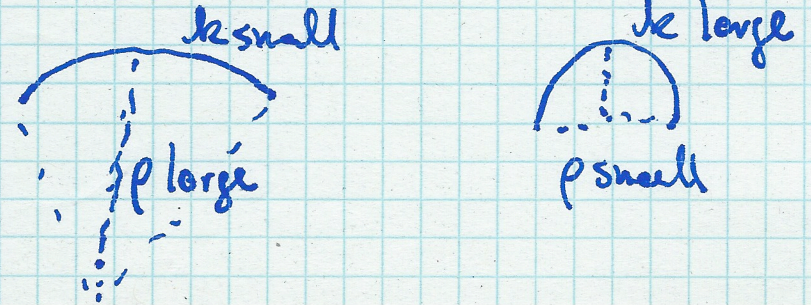
$$\frac{d\hat{T}}{ds} = \frac{d\hat{T}}{d\theta}\frac{d\theta}{ds} = \left[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right]\frac{d\theta}{ds}$$

$$\therefore \left|\frac{d\hat{T}}{ds}\right| \equiv k = \left|\frac{d\theta}{ds}\right|$$

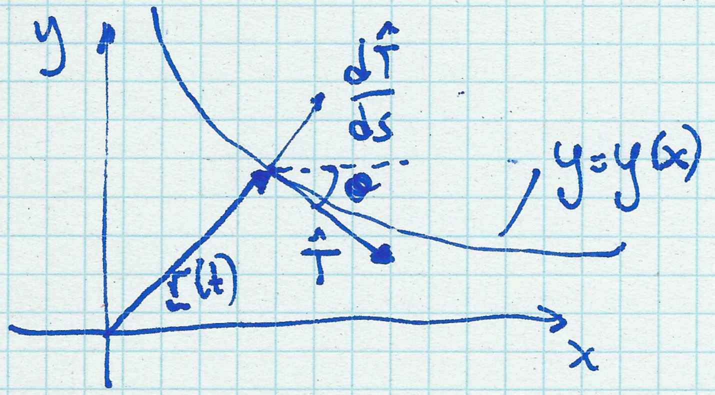
k measures the absolute rate of change of direction.

When k is large : bending sharply
is small : otherwise

$$\rho \equiv \frac{1}{k}$$ "radius of curvature"



(x) Derive an expression for a curve described by the function $y = y(x)$



[we want to find $\frac{d\theta}{ds}$]

know that tangent $\tan\theta = \frac{dy}{dx}$

$\underline{r}(t) = \langle x(t), y(t) \rangle$

from figure $\frac{dy}{dx} = \tan\theta = \frac{dy/dt}{dx/dt} = \frac{y'}{x'}$ (A)

and $\left[k = \frac{d\theta}{ds} = \frac{d\theta/dt}{ds/dt} = \frac{d\theta}{dt} \frac{1}{\sqrt{x'^2 + y'^2}} \right]$ (B)

so if we find $\frac{d\theta}{dt}$ in terms of x, y we're done.

Focus on (A) & differentiate wrt t:

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y'}{x'}\right)$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{y'}{x'}\right)$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{\sec^2\theta}\frac{d}{dt}\left(\frac{y'}{x'}\right) = \frac{1}{1+\tan^2\theta}\frac{d}{dt}\left(\frac{y'}{x'}\right)$$

using (A):

$$= \frac{1}{1+\frac{y'^2}{x'^2}}\frac{d}{dt}\left(\frac{y'}{x'}\right) = \frac{x'y''-y'x''}{x'^2+y'^2} \quad \text{(C)}$$

plugging (C) for $\frac{d\theta}{dt}$ in (B):

$$k = \frac{d\theta}{ds} = \frac{x'y''-y'x''}{x'^2+y'^2}\frac{1}{\sqrt{x'^2+y'^2}} \quad \therefore$$

$$\boxed{k = \frac{d^2y}{dx^2}\frac{1}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}}$$

if curve changes gently $\left(\frac{dy}{dx}\right) \ll 1$ and $k \simeq \frac{d^2y}{dx^2}$

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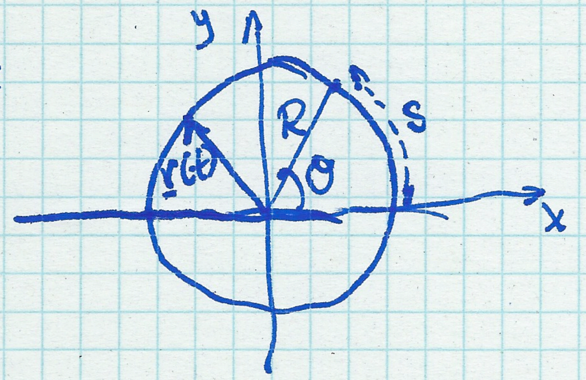
motion at CONSTANT RATE

ex) Suppose a mass m moving in a circle of radius R. Find $\underset{\sim}{r}$, $\underset{\sim}{v}$, $\underset{\sim}{a}$. Express in $(\hat{T}, \hat{N}, \hat{B})$ frame

in 2D:

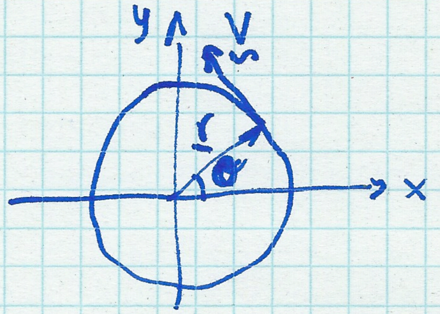


$\underset{\sim}{r}(t) = R(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$

ω is constant

$t \in \mathbb{R}$

$$\underset{\sim}{V} = \frac{dr}{dt} = +R\omega\left[-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}\right]$$



$$|\underset{\sim}{V}| = v = \sqrt{\underset{\sim}{V}\cdot\underset{\sim}{V}} = R\omega$$

so given ω the speed is $\propto R$.

it's clear that $\underset{\sim}{r}\cdot\underset{\sim}{V} = 0$ orthogonal

the arc length $s = R\theta$. But

$$\frac{ds}{dt} = \frac{d}{dt}(R\theta) = R\frac{d\theta}{dt} + \cancel{\frac{dR}{dt}}^{\,0}\theta = R\omega$$

tangential speed

angular speed

$$\boxed{v = R\omega}$$

$\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} = -R\omega^2 \underset{\sim}{r}$ so pointing to origin.

$$\underset{\sim}{a} = a_T \hat{T} + a_N \hat{N} = \frac{dv}{dt}\hat{T} + kv^2 \hat{N}$$

$$a_T = \frac{dv}{dt} = \frac{d}{dt}(\cancel{R\omega})^{0}$$ so zero tangential acceleration

<u>Recall</u> $s = R\theta$:

$$k = \frac{d\theta}{ds} = \frac{d}{ds}\left(\frac{s}{R}\right) = \frac{1}{R}$$

$$\therefore a_N = kv^2 = \frac{1}{R}R^2\omega^2 = R\omega^2$$ "centripetal acceleration"

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$$\hat{T} = \frac{\underset{\sim}{v}}{v} \qquad \hat{N} = \frac{\frac{d\underset{\sim}{T}}{ds}}{\left|\frac{d\underset{\sim}{T}}{ds}\right|}$$

$$\hat{T} = \frac{1}{R\omega}\left[-R\omega \sin\omega t\,\hat{i} + R\cos\omega t\,\hat{j}\right] = -\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}$$ tangent

$$\frac{d\hat{T}}{ds} = \frac{d\hat{T}}{dt}\frac{dt}{ds} = \frac{1}{v}\frac{d\hat{T}}{dt} = \frac{1}{R\omega}\frac{d\hat{T}}{dt} = -\frac{1}{R}\left[\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\right]$$

$$\therefore \underset{\sim}{N} = -\left[\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\right] = \frac{-\underset{\sim}{r}}{|\underset{\sim}{r}|}$$

pointing to origin.

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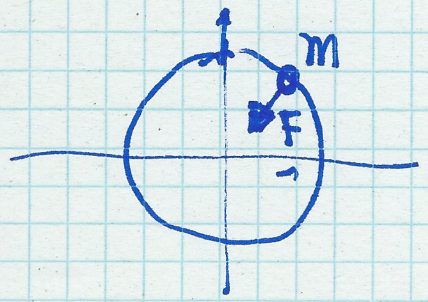
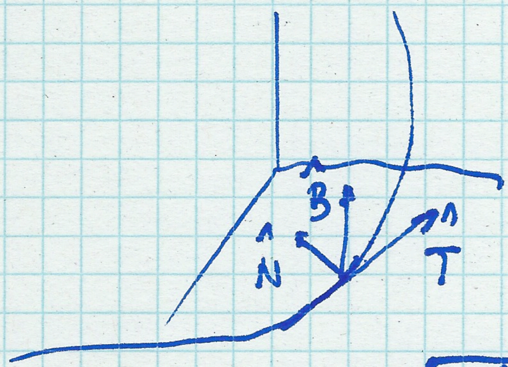


$$|F| = \frac{mv^2}{R}$$

is centripetal force.

Back to $\hat{B} = \hat{T} \times \hat{N}$ so if you found any 2 you can find the third.



We would like (Frenet-Serret)

$$\frac{d\hat{T}}{ds} = \cdots \qquad \frac{d\hat{B}}{ds} = \cdots$$

$$\frac{d\hat{N}}{ds} = \cdots$$

we found previously that $\boxed{\frac{d\hat{T}}{ds} = k\hat{N}}$ (*)

$$\frac{d\hat{B}}{ds} = \frac{d}{ds}(\hat{T} \times \hat{N}) = \frac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \quad \text{by } (*)$$

$$= \cancel{(k\hat{N}) \times \hat{N}} + \hat{T} \times \frac{d\hat{N}}{ds}$$

note: since $|\hat{B}| = 1 \Rightarrow \left.\begin{array}{l} \frac{d\hat{B}}{ds} \perp \hat{B} \\ \frac{d\hat{B}}{ds} \perp \hat{T} \end{array}\right\} \therefore \frac{d\hat{B}}{ds} \parallel \hat{N}$

$$\boxed{\therefore \frac{d\hat{B}}{ds} = -\tau(s)\hat{N}} \quad (**)$$

"torsion" of curve

Need to find $\frac{d\hat{N}}{ds}$ to complete set:

$$\frac{d\hat{N}}{ds} = \frac{d}{ds}\left(\hat{B}\times\hat{T}\right) = \frac{d\hat{B}}{ds}\times\hat{T} + \hat{B}\times\frac{d\hat{T}}{ds} \quad \text{by } (*) \,\&\, (**)$$

$$= -\tau(s)\hat{N}\times\hat{T} + \hat{B}\times k\hat{N} = \tau(s)\hat{B} - k(s)\hat{T} \quad (***)$$

Combining $(*)-(***)$ The Frenet Serret Equations

$$\begin{cases} \dfrac{d\hat{T}}{ds} = k\hat{N} \\[2ex] \dfrac{d\hat{B}}{ds} = -\tau(s)\hat{N} \\[2ex] \dfrac{d\hat{N}}{ds} = \tau(s)\hat{B} - k(s)\hat{T} \end{cases}$$

or $\dfrac{d\hat{X}}{ds} = A(s)\hat{X}$ $\qquad \hat{X} = \begin{bmatrix} \hat{T} \\ \hat{B} \\ \hat{N} \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & -\tau \\ -k & \tau & 0 \end{bmatrix}$$

$$\hat{T} = \frac{\underline{r}'}{|\underline{r}'|} \qquad \hat{N} = \frac{\hat{T}'}{|\hat{T}'|} = \frac{\underline{r}'\times(\underline{r}''\times\underline{r}')}{|\underline{r}'|\,\|\underline{r}''\times\underline{r}'\|} \qquad \hat{B} = \hat{T}\times\hat{N} = \frac{\underline{r}''\times\underline{r}'}{\|\underline{r}''\times\underline{r}'\|}$$

$$k = |r''\times r'|/|r'|^3$$