11.7 Motion in Space

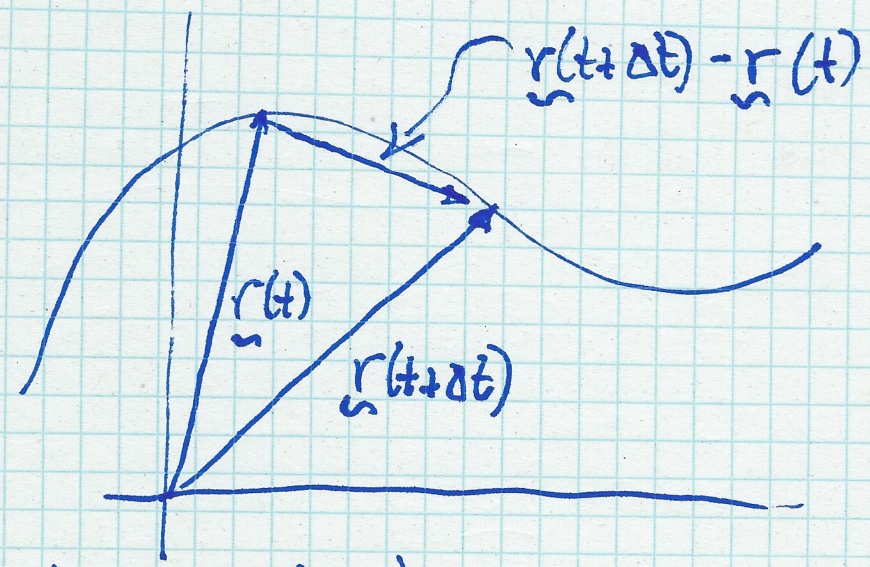
3D but the following applies to N-Dimensions

Let $\underset{\sim}{r}(t) = \langle x(t), y(t), z(t) \rangle$

"Displacement vector"

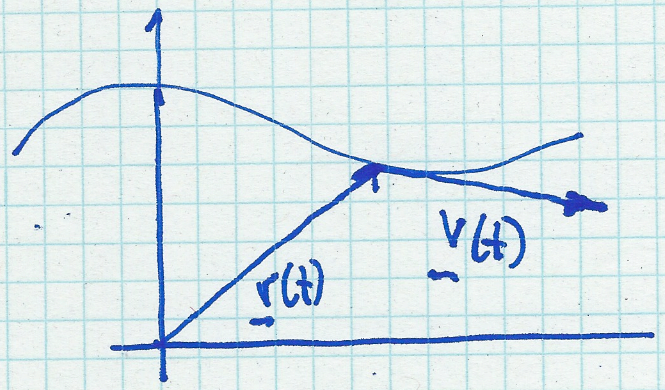
The "velocity vector" is

$$\underset{\sim}{v} \equiv \frac{d\underset{\sim}{r}}{dt} = \lim_{\Delta t \to 0} \frac{\underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t)}{\Delta t} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$



The "acceleration vector"

$$\underset{\sim}{a} \equiv \frac{d\underset{\sim}{v}}{dt} = \lim_{\Delta t \to 0} \frac{\underset{\sim}{v}(t+\Delta t) - \underset{\sim}{v}(t)}{\Delta t} = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right\rangle$$



$\underset{\sim}{v}$ is always tangent to path traced out by $\underset{\sim}{r} = \langle x(t), y(t), z(t) \rangle$

$$\underset{\sim}{v} = v\,\hat{e}_v \quad , \quad |\underset{\sim}{v}| \equiv v \text{ "the speed"}$$

$$\underset{\sim}{v} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$\hat{e}_v$ is unit vector pointing in the direction of $\underset{\sim}{v}$

$$|\underset{\sim}{v}| = \sqrt{\underset{\sim}{v}\cdot\underset{\sim}{v}} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\underset{\sim}{r}(t) = \int_{t_0}^{t} \underset{\sim}{v}(s)\,ds = \int^{t} \underset{\sim}{v}(s)\,ds + \underset{\sim}{r}(t_0)$$

$$\underset{\sim}{v}(t) = \int_{t_0}^{t} \underset{\sim}{a}(s)\,ds = \int^{t} \underset{\sim}{a}(s) + \underset{\sim}{v}(t_0)$$

ex) **2D** let $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle 2\cos t, 2\sin t \rangle$

$t \geq 0$

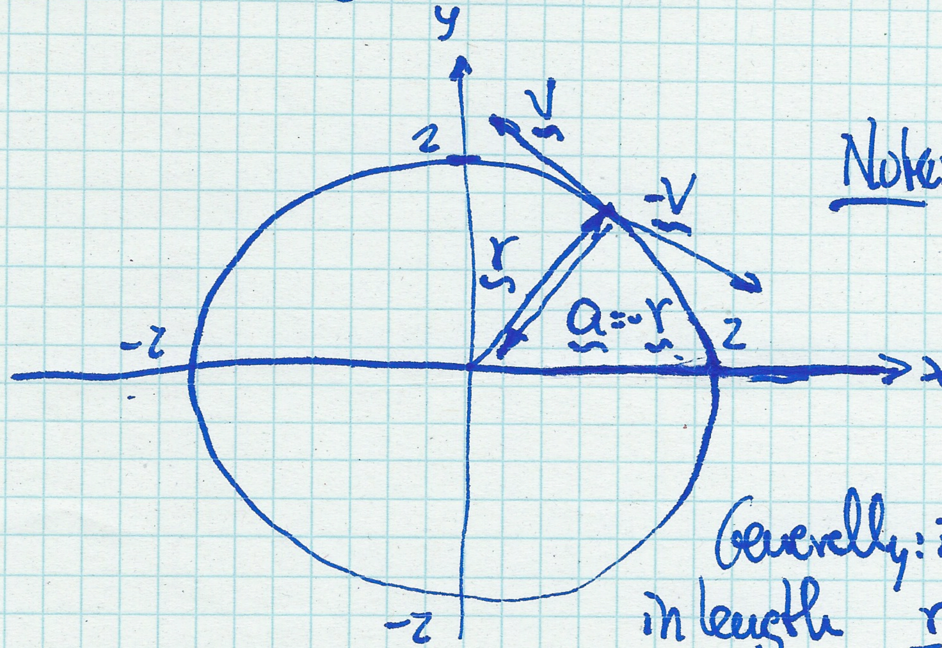
$$\mathbf{v} = \mathbf{r}'(t) = \langle -2\sin t, 2\cos t \rangle, \quad t \geq 0$$

$$v = |\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$$

$$\hat{e}_v \equiv \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{2}\langle -2\sin t, 2\cos t \rangle = \langle -\sin t, \cos t \rangle$$

$$\therefore \mathbf{v} = v\,\hat{e}_\theta = 2\langle -\sin t, \cos t \rangle$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \langle -2\cos t, -2\sin t \rangle = -\mathbf{r}$$ (in this special case!)



Note: $\frac{d}{dt}|\mathbf{r}|^2 = \frac{d}{dt}\mathbf{r}\cdot\mathbf{r}$

$= 2\mathbf{v}\cdot\mathbf{r} = 0$

$\therefore \mathbf{v} \perp \mathbf{r}$

Generally: if $\mathbf{r}$ is constant in length $\mathbf{r}\cdot\mathbf{v} = 0$

Newtonian Mechanics

2nd Law

$$\frac{d}{dt}(m\underset{\sim}{v}) = \sum \underset{\sim}{F} \quad \text{"sum of forces"}$$

assume mass m is constant.

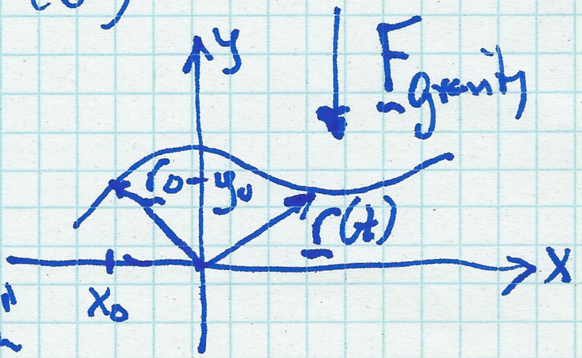
$$(¥) \quad m\underset{\sim}{a} = \sum \underset{\sim}{F}$$

Motion in Gravitational Field

Assume 2D and gravity force is constant g

$$F_{gravity} = mg \qquad \text{so } (¥) \text{ is}$$

$$(\ni) \quad m\underset{\sim}{r}''(t) = \langle 0, -mg \rangle$$



let $\underset{\sim}{r}_0 = \langle x_0, y_0 \rangle$ "initial position"

$$\underset{\sim}{r}_0 = \langle x(t_0), y(t_0) \rangle = \langle x_0, y_0 \rangle$$

so time $t = t_0$ is "initial time"

$$\underset{\sim}{v}_0 = \langle u(t_0), w(t_0) \rangle \text{ is "initial velocity"}$$

Goal: Given some initial conditions, find $\underset{\sim}{r}(t)$, $\underset{\sim}{v}(t)$, $\underset{\sim}{a}(t)$ for $t \geq t_0$

From (7) $\quad \underset{\sim}{r}''(t) = \langle 0, -g \rangle$

$$\therefore \underset{\sim}{v} = \int_{t_0}^{t} \underset{\sim}{r}''(s)\,ds = \underset{\sim}{v}_0 + \int^{t} \langle 0, -g \rangle\, ds$$

$$\underset{\sim}{v} = \underset{\sim}{v}_0 + \langle 0, -gt \rangle = \langle u_0, w_0 - gt \rangle$$

$$\underset{\sim}{r} = \int_{t_0}^{t} \underset{\sim}{v}(s)\,ds = \underset{\sim}{r}_0 + \int^{t} \underset{\sim}{v}_0\, ds + \int^{t} \langle 0, -gs \rangle\, ds$$

$$\underset{\sim}{r} = \underset{\sim}{r}_0 + \underset{\sim}{v}_0 t + \langle 0, -g\tfrac{t^2}{2} \rangle$$

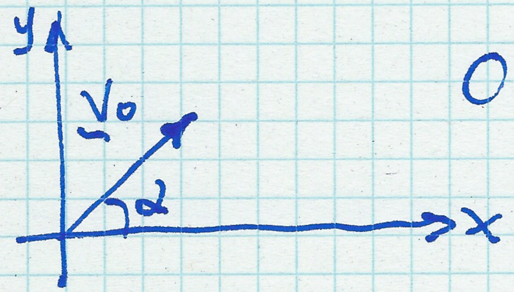
$$\underset{\sim}{r} = \langle x_0 + u_0 t,\ y_0 + w_0 t - \tfrac{1}{2} g t^2 \rangle$$

ex) Projectile Motion (point mass, no drag)

Setup: gravity constant $\sim 10\,m/s^2 \sim 32\,ft/sec^2 \equiv g$

wlog (without loss of generality) $\begin{cases} \text{take } \underset{\sim}{r}_0 = \langle 0, 0 \rangle \\ t_0 = 0 \end{cases}$

$\underset{\sim}{v}(0) = \underset{\sim}{v}_0 = \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle$; $v_0 = |\underset{\sim}{v}_0|$



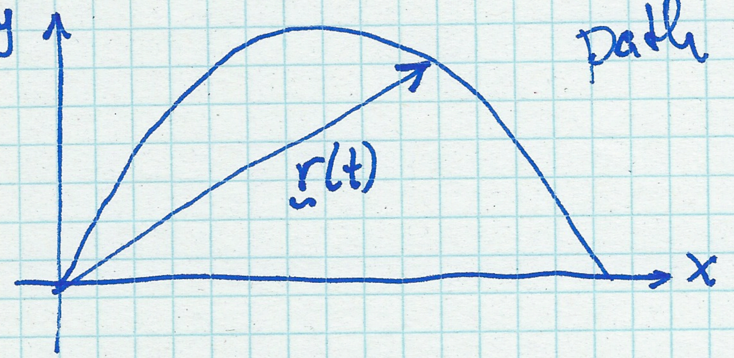
$0 \le \alpha \le \frac{\pi}{2}$

For $t > 0$

$\underset{\sim}{r}''(t) = \langle 0, -g \rangle$ (a)

$\underset{\sim}{r}'(t) = \langle v_0 \cos\alpha, v_0 \sin\alpha - gt \rangle$ (b)

$\underset{\sim}{r}(t) = \langle v_0 \cos\alpha\, t, v_0 \sin\alpha\, t - \frac{1}{2} g t^2 \rangle$ (c)



path is a parabola

Q: find "time of flight": means find t_f such that

$y(t_f) = 0$ with $t_f > 0$. From (c)

$$v_0 \sin\alpha \, t_f = \frac{1}{2} g t_f^2 \qquad t_f > 0$$

$$\therefore \; t_f = \frac{2 v_0 \sin\alpha}{g}$$

Q: find "range": means find $x(t_f)$. From (c)

$$x(t_f) = v_0 \cos\alpha \left[\frac{2 v_0 \sin\alpha}{g}\right] = \frac{2 v_0^2}{g} \cos\alpha \sin\alpha$$

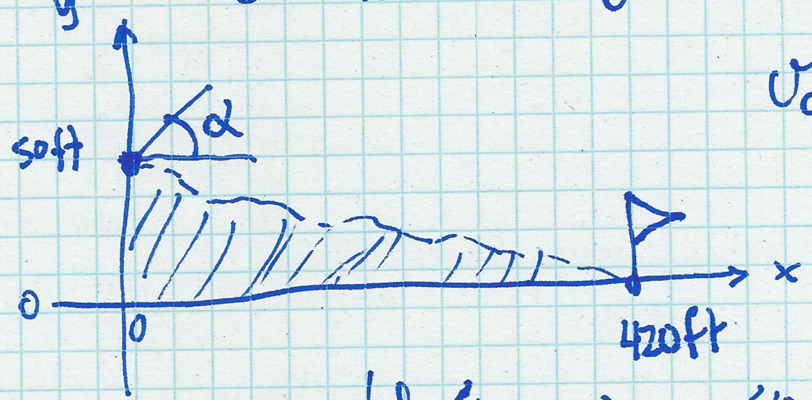
Q: Find "maximum height": means find $y'(t) = 0$ $t > 0$

from (b) $y'(t) = v_0 \sin\alpha - gt = 0 \Rightarrow t = \dfrac{v_0 \sin\alpha}{g}$

from (c) $$v_0 \sin\alpha \, \frac{v_0 \sin\alpha}{g} - \frac{1}{2} g \left(\frac{v_0 \sin\alpha}{g}\right)^2 = \frac{v_0^2 \sin^2\alpha}{2g}$$

ex) Worked example 11.7 #59

Golf: find α angle to hit ball into hole, time of flight t^*



$v_0 = 120$ ft/s

let $\langle x_0, y_0 \rangle = \langle 0, 50 \rangle$ ft

$\langle u_0, w_0 \rangle = 120 \langle \cos\alpha, \sin\alpha \rangle$ ft/s

$\underset{\sim}{a} = \langle 0, -32 \rangle$ ft/s^2; $\underset{\sim}{v} = 120\langle \cos\alpha, \sin\alpha \rangle - 32t$ ft/s,

~~Time of flight~~ $\underset{\sim}{r} = \langle 120\cos\alpha\, t,\ 50 + 120\sin\alpha\, t - \frac{1}{2}32t^2 \rangle$ ft

Time of flight t^*: $\langle 420, 0 \rangle = \underset{\sim}{r}(t^*)$

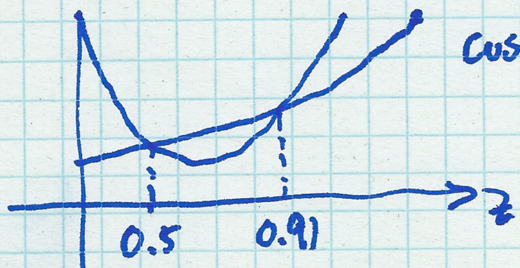
$$\begin{cases} 16t^{*2} - 120\sin\alpha\, t^* - 50 = 0 & (**) \\ 120\cos\alpha\, t^* = 420 \Rightarrow t^* = \dfrac{21}{\cos\alpha} & (***) \end{cases}$$

substitute (***) in (**)

$$\cancel{\cos^2\alpha + \frac{1260}{25}} \cancel{-35} \quad \cos^2\alpha + \frac{1260}{25}\cos\alpha\sin\alpha - \frac{3528}{25} = 0$$

To find α: let $z = \cos\alpha$ $\therefore \sin\alpha = \sqrt{1-z^2}$; $a = \frac{1260}{25}$, $b = \frac{3528}{25}$

$$z^2 + a\sqrt{1-z^2}\, z - b = 0 \text{ or } z^2 - 1.2005 = 8.4z\sqrt{1-z^2}$$



$\cos\alpha \begin{cases} 0.5 \\ \text{or} \\ 0.91 \end{cases} \Rightarrow \alpha = \begin{cases} 23.5° \\ 59.6° \end{cases}$