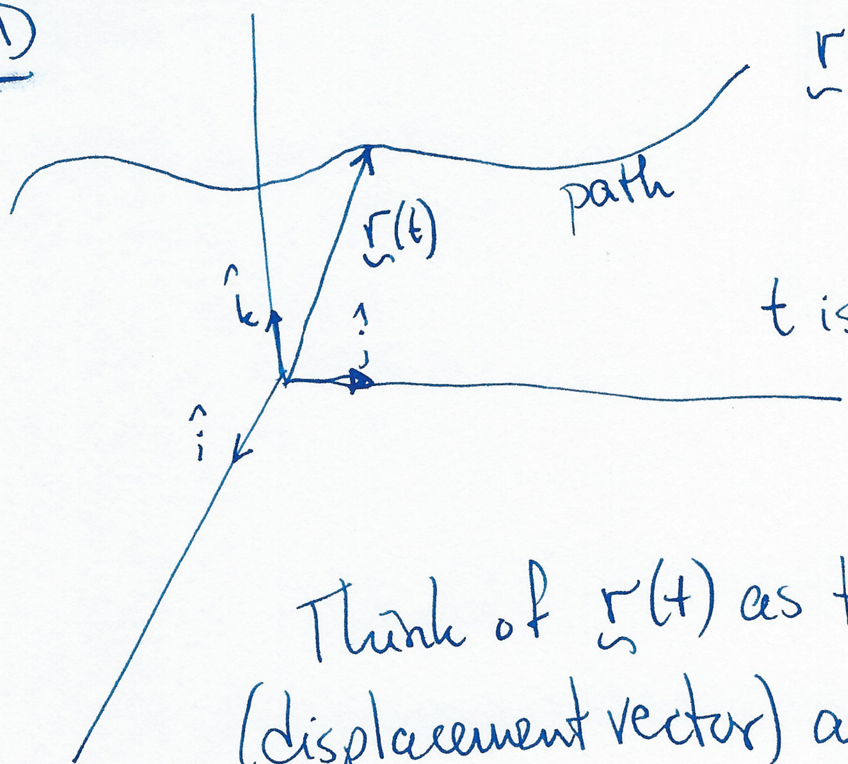
11.5 Lines & Curves in Space

3D



vector-valued function

$\underset{\sim}{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$= \langle x(t), y(t), z(t) \rangle$

t is a parameter

Think of $\underset{\sim}{r}(t)$ as the position (displacement vector) and t as time (parameter that takes on a specified set of values).

ex) A straight line or path

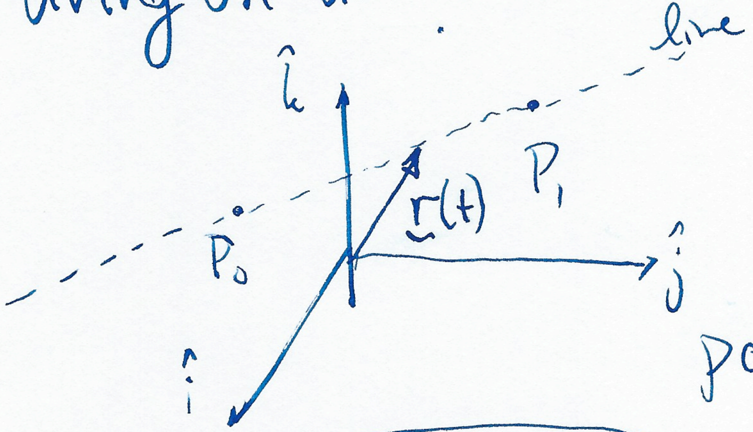
in 2D $\begin{cases} \underset{\sim}{r}(t) = \langle x(t), y(t) \rangle \\ \text{or } \underset{\sim}{r}(x,y) = \langle x, y(x) \rangle \quad (\star) \end{cases}$

In $(\star)$ $y = mx + b$ describing a straight line by a LINEAR relation.

To find line need minimally $\left\{ \begin{array}{l} \text{2 points in Space} \\ \quad\text{or} \\ \text{1 point \& direction} \end{array} \right.$

Take $P_0 = (x_0, y_0, z_0)$ & $P_1 = (x_1, y_1, z_1)$

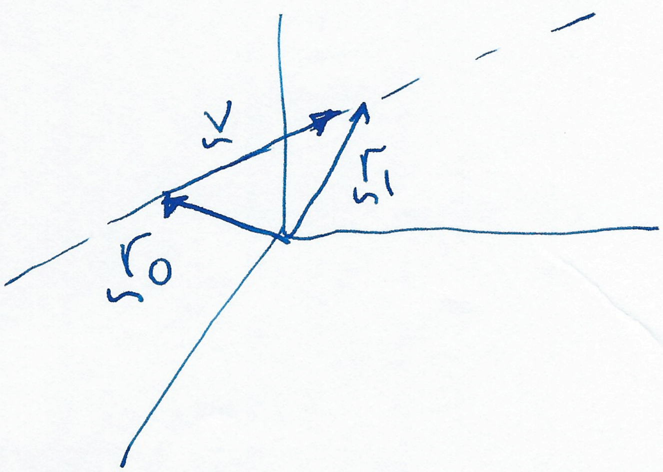
living on line:



$\underset{\sim}{r}(t)$ for $t \in (-\infty, \infty)$ will be made to trace straight line passing through P_0 & P_1

$$\boxed{(\$) \quad \underset{\sim}{r}(t) = \underset{\sim}{r}_0 + \underset{\sim}{v}\, t, \quad t \in \mathbb{R}^1}$$

, $\underset{\sim}{v}$ is a constant vector aligned with straight line. $\underset{\sim}{r}_0$ is a vector whose tip is on straight line & is constant



let $\underset{\sim}{r}_0 = \langle x_0, y_0, z_0 \rangle$

$\underset{\sim}{r}_1 = \langle x_1, y_1, z_1 \rangle$

$\underset{\sim}{V} = \underset{\sim}{r}_1 - \underset{\sim}{r}_2$

$= \langle a, b, c \rangle$

$\therefore$ ($) is $\underset{\sim}{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

$t \in \mathbb{R}^1$

ex) (a) Find displacement vector for line passing through $P_0 = (-3, 5, 8)$ $P_1 = (4, 2, -1)$

Refer to ($): $\underset{\sim}{v} = \langle a, b, c \rangle = \underset{\sim}{r}_1 - \underset{\sim}{r}_0 = \langle 7, -3, -9 \rangle$

$\underset{\sim}{r}_0 = \langle -3, 5, 8 \rangle$ $\underset{\sim}{r}_1 = \langle 4, 2, -1 \rangle$

$\therefore \underset{\sim}{r}(t) = \underset{\sim}{r}_0 + \langle 7, -3, -9 \rangle t$ (*) $t \in \mathbb{R}^1$

(b) Find line <u>segment</u> between P_0 & P_1

Using (*) and setting $t = 0$ we get $\underset{\sim}{r}(t)$ to agree with $\underset{\sim}{r}_0$. Need to find t^* s.t. $\underset{\sim}{r}(t^*) = \underset{\sim}{r}_1$.

$\underset{\sim}{r}(t^*) = \langle 4, 2, -1 \rangle$ so $t^* = 1$

$\therefore \underset{\sim}{r}(t) = \underset{\sim}{r}_0 + \langle 7, -3, -9 \rangle t$ $t \in [0, 1]$

(c) Find equation for projection of (*) onto the xz-plane.

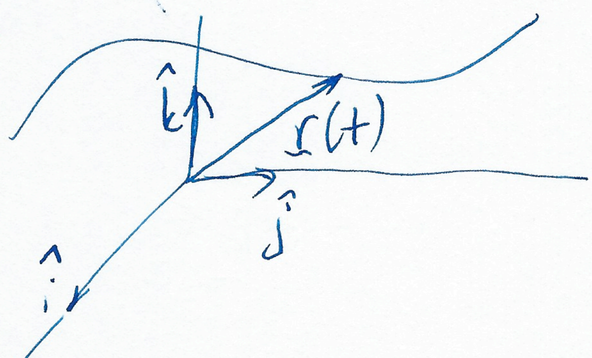
Set $y=0$ in (*) $\begin{cases} x = -3+7t & ① \\ z = 8-9t & ② \end{cases}$

eliminating t from ① & ② we get $x = \dfrac{29-7z}{9}$

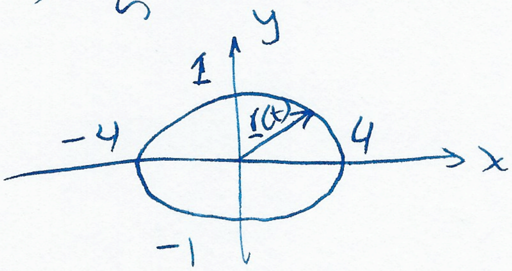
Curves in Space

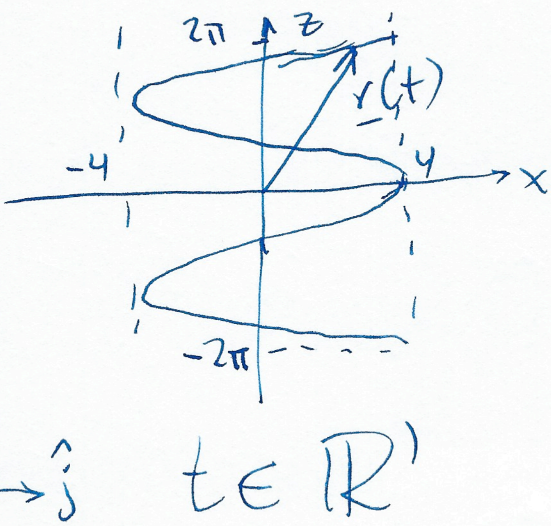
$\underset{\sim}{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

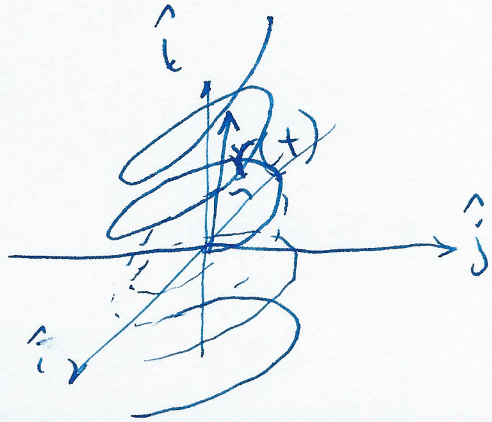
$a \le t \le b$



ex) $\underset{\sim}{r}(t) = 4\cos t\,\hat{i} + \sin t\,\hat{j} + \dfrac{t}{2\pi}\hat{k}$ (a Helix)







$t \in \mathbb{R}$

Limits & Continuity

$$\lim_{t \to a} \underset{\sim}{r}(t) = \underset{\sim}{L} \quad \text{provided} \quad \lim_{t \to a} |\underset{\sim}{r}(t) - \underset{\sim}{L}| = 0$$

ex) Find limit as $t \to 0$ of

$$\underset{\sim}{r}(t) = e^t \hat{i} + \cos \pi t \hat{j} + e^{-t} \hat{k} \quad (\#)$$

$$\lim_{t \to 0} \underset{\sim}{r}(t) = \underset{\sim}{L} = \hat{i} + \hat{j} + \hat{k}$$

ex) Find limit as $t \to 0$ of $\underset{\sim}{r}(t) = \frac{\sin t}{t} \hat{i} + 5\hat{k}$

$$\hat{i} \cdot \underset{\sim}{r}(t) = \frac{\sin t}{t} \qquad \lim_{t \to 0} \frac{\sin t}{t} = \lim_{t \to 0} \frac{\cos t}{1} = 1$$

(L'Hopital's)

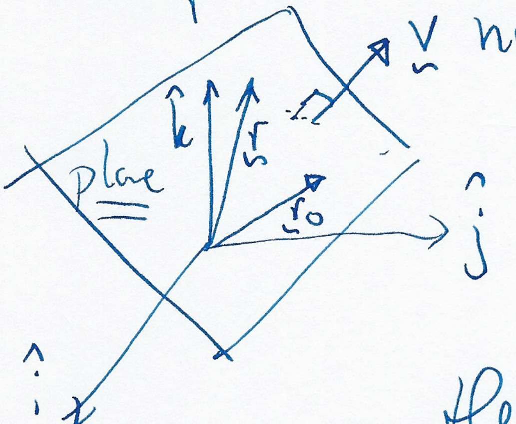
$$\therefore \lim_{t \to 0} \left\langle \frac{\sin t}{t}, 0, 5 \right\rangle = \langle 1, 0, 5 \rangle$$

ex) Find $\lim_{t \to \infty} \underset{\sim}{r}(t)$ with $\underset{\sim}{r}(t)$ given by $(\#)$

Note: $\lim_{t \to \infty} \hat{i} \cdot \underset{\sim}{r}(t) = 0$, $\lim_{t \to \infty} \hat{j} \cdot \underset{\sim}{r}(t)$ is undefined

$\lim_{t \to \infty} \hat{k} \cdot r(t)$ is undefined $\therefore \lim_{t \to \infty} \underset{\sim}{r}(t)$ is undefined

Equation of a Plane (in 3D)



$\mathbf{v}$ normal to plane

If $\mathbf{r}$'s tip as well as $\mathbf{r}_0$'s tip live on plane

then $\mathbf{r} - \mathbf{r}_0$ is a vector that lives on plane.

$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$

$\mathbf{v} = \langle a, b, c \rangle$

If $\mathbf{v}$ is normal to plane then

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{v} = 0$$

or $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$