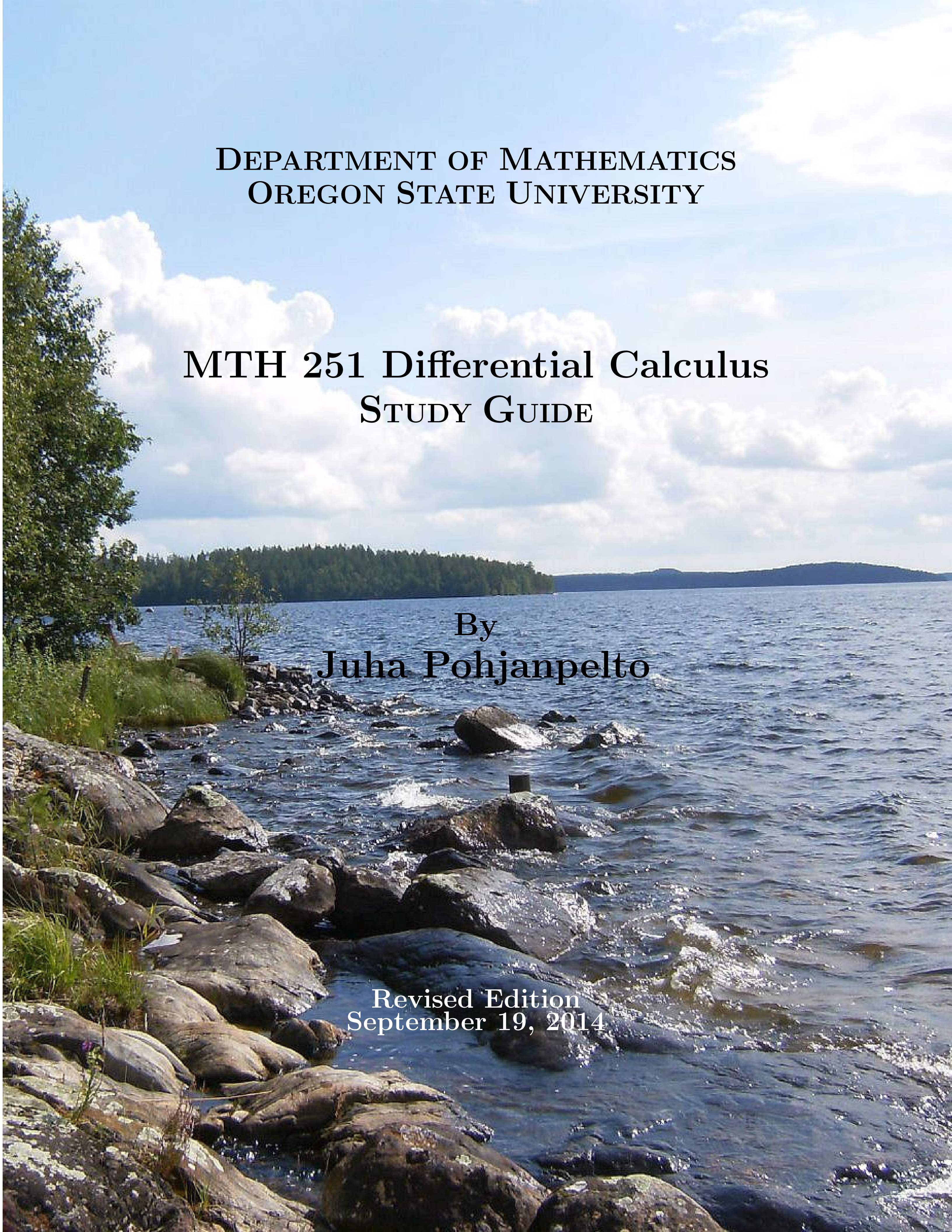

Department of Mathematics
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MTH 251 Differential Calculus
Study Guide

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Contents

SYLLABUS FOR MTH 251

Study Guide	Text	Topic
Lesson 0	Ch. 1	Functions
Lesson 1	§2.1	Idea of Limits
Lesson 2	§2.2	Definitions of Limits
Lesson 3	§2.3	Techniques for Computing Limits
Lesson 4	§2.4	Infinite Limits
Lesson 5	§2.5	Limits at Infinity
Lesson 6	§2.6	Continuity
Lesson 7	§3.1, 3.2	Introducing and Working with Derivatives
Lesson 8	§3.3	Rules of Differentiation
Lesson 9	§3.4	The Product and Quotient Rules
Lesson 10	§3.5	Derivatives of Trigonometric Functions
		Catch-up/Review
Lesson 11	§3.6	Derivatives as Rates of Change
Lesson 12	§3.7	The Chain Rule
Lesson 13	§3.8	Implicit Differentiation
Lesson 14	§3.9	Derivatives of Logarithmic and Exponential Functions
Lesson 15	§3.10	Derivatives of Inverse Trigonometric Functions
Lesson 16	§3.11	Related Rates
Lesson 17	§4.1	Maxima and Minima
Lesson 18	§4.2	What Derivatives Tell Us
Lesson 19	§4.3	Graphing Functions
		Catch-up/Review
Lesson 20	§4.4	Optimization Problems
Lesson 21	§4.5	Linear Approximations and Differentials
Lesson 22	§4.6	Mean Value Theorem
Lesson 23†	§4.7	L'Hôpital's Rule
		Catch-up/Review

†Optional

Introduction and Notes for Students

Introduction. This is a study guide for MTH 251 Differential Calculus. It is intended to be used in conjunction with the second edition of the text *Calculus – Early Transcendentals* by Briggs, Cochran, and Gillett. The study guide is designed for a ten week term with 29 lectures and 9 laboratory sessions, with two class hours reserved for exams, four for review and catch-up, and 23 lessons devoted to new material from the text. The lesson plan for the course by and large follows the organization of the material in the text.

A number of recitation sessions will involve scheduled laboratory activities as designed by your instructor, and the remaining time is set aside for review and for questions and answers. *You should consult your instructor for a detailed syllabus for your section of the course.*

Comments and suggestions on study habits. In this course much new material will be presented at a rapid pace. You will also be expected to understand and apply mathematical concepts and reasoning, not merely perform calculations. Therefore developing good study habits from the outset in order to keep up with the course is particularly important. Listed below are a number of points to heed.

Time commitment: It is essential to devote enough time on a daily basis to the course. You should plan on spending at least *two to three hours* studying the material and solving assignments for each hour of lecture. If you have encountered some of this material before, it is easy to fall into the habit of not dedicating enough time to the course at the beginning of the term. Then, when more challenging topics are presented later on, you could find yourself too far behind to catch up.

Algebra skills. Applying the proper algebraic manipulations taught in precalculus courses is one of the most common hurdles for students taking a calculus class. If your skills are rusty, practice now and seek help before it will be too late. This course is taught with the premise that you master the basics of algebra and trigonometry covered in your previous mathematics courses.

Attendance. Your instructors will from time to time introduce a new viewpoint or amplify on the material set forth in the text or the study guide. Beside the calculus text, exams for the course will be based on the lectures, recitations, lab activities and assignments, and you are accountable for all the materials. It is then in your own interest to plan on attending all the class meetings and to get notes from another student for any that you might have missed.

Homework. Exercises in the text are divided into several categories. *Review Questions* test your conceptual understanding of the narrative, while solving *Basic Skills* exercises will improve your computational dexterity. Exercises under the *Further Explorations* heading are built on the Basic Skills problems and are more demanding, while exercises under *Applications* emphasize the utility of calculus in practical problems. Finally, *Additional Exercises* will challenge your thinking and often involve mathematical proofs. Each chapter in the text concludes with *Review Exercises* which will help you to synthesize the contents of the entire chapter.

Most students can only excel in this course by solving a large number of problems and you should therefore aim to work through most of the exercises in the text. As a starting point, you will find lists of basic *Starter Problems* and the more challenging *Recommended Problems* at the end of each lesson in this study guide. Keep in mind that simply getting the answer at the back of the book should not be your only goal, but that understanding the principles and methods needed to solve an exercise is the primary purpose of an assignment, and also a requisite for solving similar problems in the exams and in real-life situations.

Consult your instructor for further details on graded assignments, which may also include online homework administered via the MyLab & Mastering platform.

Laboratory Activities. There are detailed group activities presented in the Laboratory Manual of this study guide. During the term you may be assigned to work on a subset of these as part of the weekly recitation sessions. Your instructor will provide a detailed schedule for the activities.

Other Resources. The Mathematics Learning Center (MLC) provides drop-in help for all lower division mathematics courses. The MLC is located on the ground floor of Kidder Hall in room 108, and is normally open Monday through Thursday from 9 a.m. to 5 p.m. and on Fridays from 9 a.m.to 4 p.m., from the second week of each term through the dead week. The MLC also provides evening tutoring at the Collaborative Learning Center in the Valley Library. Current hours can be found at the MLC home page.

By purchasing the course text you will gain access to MyLab & Mastering, an online calculus portal maintained by the publisher. In addition to homework problems and tests with automated grading, the site offers a number of useful tools, including PowerPoint and video lectures, review cards, and tutorial exercises, for organizing your studies and to facilitate learning the course material.

Acknowledgments. This study guide relies on the previous Mathematic Department's MTH 251 study guides, with material written by Bob Burton, Tevian Dray, Christine Escher, and Dennis Garity. Valuable comments were also received from Dianne Hart, Peter Argyres, and Stephen Scarborough. To all of these people I extend my heartfelt thanks.

MTH 251 Sample Symbolic Differentiation Test

NAME:________________________________ Student ID#: __________

Show only your answers on this page. Do your work on scratch paper and turn in your scrap paper with this page. You must show your work to receive full credit. NO CALCULATORS OR NOTES ARE ALLOWED!

Compute the following derivatives: (You do not need to simplify your answers.)

1. $\frac{d}{dx}(2x - 4x^4 - 16)$

2. $\frac{d}{dx}\frac{x^3}{x^2+3}$

3. $\frac{d}{dx}\sqrt{\cos x + 2}$

4. $\frac{d}{dx}\sec^3 x$

5. $\frac{d}{dx}\sin(x^3 - 7)$

6. $\frac{d}{dx}(x^5 - 14)^{4/3}$

7. $\frac{d}{dx}(x \tan x)$

8. $\frac{d}{dx}\frac{\cos x}{x+1}$

9. $\frac{d}{dx}\left(\frac{1}{x^3}\right)$

10. $\frac{d}{dx}\left(\sin(x)\cos(2x)\right)$

Lesson 0 – Ch. 1: Functions

The chapter contains a summary of some of the background material required for this course. You should carefully review it on your own as it will not be thoroughly covered in class.

Section 1.1 Browse through the section to ensure that your understand the basic concepts and are able to solve the exercises in the text. You should be familiar with the following terminology:

- Domain of a function
- Range of a function
- Independent variable
- Dependent variable
- Graph of a function
- Vertical line test
- Composite function
- Even function
- Odd function
- Secant line

An *even function* satisfies $f(-x) = f(x)$ for all x in the domain of $f(x)$, while an *odd function* is characterized by $f(-x) = -f(x)$. For example, any even powered monomial $f(x) = x^{2n}$, $n = 1, 2, 3, \dots$, yields an even function, and any odd powered monomial $f(x) = x^{2n+1}$, $n = 0, 1, 2, \dots$, an odd function. Also, as you can easily verify, the absolute value function $|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0, \end{cases}$ is even.

Section 1.2 treats various types of elementary functions and their graphs. For you the most important ones to be able to work with are

- Polynomials
- Rational functions
- Algebraic functions
- Exponential functions
- Logarithmic functions
- Trigonometric functions

A *linear function* $f(x) = mx + b$, where m, b are constants, is a polynomial function of *degree* 1. The *square root* function $f(x) = \sqrt{x}$ and the *cubic root* function $f(x) = x^{1/3}$ are familiar examples of algebraic functions.

The *slope function* $g(x)$ of a function $f(x)$ specifies the slope of the graph of $y = f(x)$ at x. You will discover in this class that, for example, the slope function of the parabola $y = x^2$ is $g(x) = 2x$, while the slope function of the sine function $y = \sin x$ is $g(x) = \cos x$.

You can construct new functions by *shifting* and *scaling* the graph of a given function $f(x)$ in the x and y-directions. As an example, graph (by hand!) the x-shift $\sin(x+2)$, the y-shift $\sin x+2$, the x-scaling $\sin(2x)$, and the y-scaling $2\sin x$ of the sine function in the same grid. Make sure you understand how the graphs of these are related to the graph of their progenitor!

Section 1.3. Brush up on inverse functions and the horizontal line test for checking whether a function is one-to-one. As an exercise, determine the largest intervals on which the function $f(x) = x^2 - 2x$ is one-to-one and find its inverse function on each of the intervals you have identified.

Review the definition of the exponential function and the logarithmic function as its inverse. You will be expected to understand their basic properties and to be able to apply the *exponent* and *logarithm rules* as spelled out in the margin of the text.

Section 1.4. Start by reviewing the definition of the radian measure and, for practise, express the angle measurements 15°, 30°, 45°, 60°, 90°, 180° in radians.

You should be familiar with the graphs of the basic trigonometric functions

$$\sin\theta, \quad \cos\theta, \quad \sec\theta, \quad \csc\theta, \quad \tan\theta, \quad \cot\theta,$$

and to be able to find their precise values for the angles

$$\theta = 0, \qquad \pi/6, \qquad \pi/4, \qquad \pi/2, \qquad \pi,$$

without a calculator. The various *trigonometric identities* on p. 41 of the text often prove handy in simplifying complicated trigonometric expressions.

The *inverse trigonometric functions* will be covered in Lesson 15.

Starter Problems:

Section 1.1: 6, 13, 26, 37, 42, 49, 71.

Section 1.2: 3, 7, 8, 15, 20, 29, 37.

Section 1.3: 4, 8, 10, 12, 16, 22, 31, 41, 59.

Section 1.4: 2, 3, 7, 16, 18, 25, 41.

Recommended Problems:

Section 1.1: 8, 10, 14, 18, 24, 29, 33, 39, 45, 54, 55, 76, 94.

Section 1.2: 10, 28, 32, 42, 46, 53, 63.

Section 1.3: 11, 17, 27, 28, 30, 34, 36, 44, 52, 55, 64.

Section 1.4: 4, 8, 20, 21, 34, 36, 43, 84, 96.

Lesson 1 – §2.1: The Idea of Limits

The *limit* of a function, which describes the behavior of the function near a fixed point, is a key concept in calculus. Roughly, the limit is the number that the values of a function approach as the input gets close, but is not equal, to the fixed point.

In this section the importance of limits is underscored by the way of examples involving average and instantaneous velocities, and the secant and tangent lines. You will revisit these type of problems after learning about the derivative of a function.

Starter Problems: 3, 4, 7, 15.

Recommended Problems: 6, 13, 21, 26, 29, 32.

Lesson 2 – §2.2: Definitions of Limits

Intuitively, the limit of a function $f(x)$ at $x = a$ equals L,

$$\lim_{x \to a} f(x) = L,$$

if the values of $f(x)$ are arbitrarily close to L when x is sufficiently close, but not equal, to a. Hence the limit of a function may exist at $x = a$ even when the function is not defined at $x = a$.

You can find the precise ϵ-δ definition of a limit in Section 2.7 of the text. However, the description given in Section 2.2 will by and large suffice for the purposes of this course, although you might want to study figures 2.60 – 2.62 in section 2.7 to gain a deeper understanding of the limit concept.

When computing the limit of a function at $x = a$, one considers values of $f(x)$ on both sides of a. Thus the limit of a function is often referred to as a two-sided limit. A useful variation to the basic limit concept is that of *one-sided* limits,

$$\lim_{x \to a^+} f(x), \qquad\qquad \lim_{x \to a^-} f(x),$$

where, in the first instance, one only considers values of x that are larger than a and, in the second one, smaller than a.

The limit $\lim_{x \to a} f(x)$ exists if and only if the one-sided limits $\lim_{x \to a^+} f(x)$, $\lim_{x \to a^-} f(x)$ exist and are equal. Hence a one-sided limit may exist at a point even when the two-sided limit does not. (Can you construct an example?)

You can often estimate the limit $\lim_{x \to a} f(x)$ by zooming in on the graph of $f(x)$ near $x = a$. However, graphing utilities can also lead you awry due to rounding errors as you will discover in Laboratory Activity I.

Starter Problems: 2, 5, 6, 7, 11, 21, 37.

Recommended Problems: 10, 13, 14, 17, 25, 28, 33, 41, 47.

Lesson 3 – §2.3: Techniques for Computing Limits

In this lesson you will learn a number of useful rules for computing limits for the most common types of functions you will encounter in this course. You should study the examples in the text carefully as these will help you understand how to apply the *Limit Laws* found on p. 70 of the text.

The basic rules for computing limits are as follows: Suppose that c is a constant, n a positive integer, and that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim\limits_{x\to a} c = c.$

2. $\lim\limits_{x\to a} x = a.$

3. $\lim\limits_{x\to a} (cf(x)) = c \lim\limits_{x\to a} f(x).$

4. $\lim\limits_{x\to c} (f(x) + g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x).$

5. $\lim\limits_{x\to a} (f(x)g(x)) = (\lim\limits_{x\to a} f(x))(\lim\limits_{x\to a} g(x)).$

6. $\lim\limits_{x\to a} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$, provided that $\lim\limits_{x\to a} g(x) \neq 0.$

7. $\lim\limits_{x\to a} (f(x))^n = \left(\lim\limits_{x\to a} f(x)\right)^n.$

8. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$, provided that $f(x) \geq 0$ for x near a for n even.

As an example, if $\lim\limits_{x\to 2} f(x) = -3$ and $\lim\limits_{x\to 2} g(x) = 2$, it then follows from limit laws 3, 4, and 7 that

$$\lim_{x\to 2} \big(f(x)^2 - 3g(x)\big) = \big(\lim_{x\to 2} f(x)\big)^2 - 3\lim_{x\to 2} g(x) = 3.$$

All the above rules also hold for one-sided limits with the obvious modifications.

The limits laws can be used to find the limits of *polynomial and rational functions*: If p and q are polynomials, then

$$\lim_{x\to a} p(x) = p(a), \qquad \lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}, \quad \text{provided that } q(a) \neq 0.$$

The computation of a limit $\lim_{x\to a} f(x)/g(x)$ of the quotient of two functions, where $g(a) = 0$, often calls for algebraic manipulations. Two basic techniques, canceling common factors and multiplying by the algebraic conjugate, are illustrated in Example 6 in the text.

Yet another technique for finding limits is afforded by the *Squeeze Theorem:* If $f(x) \leq g(x) \leq h(x)$ for all x near a (except possibly at a), and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then

$$\lim_{x\to a} g(x) = L.$$

You should try to decipher the content of the Squeeze Theorem in terms of the graphs of $f(x)$, $g(x)$, and $h(x)$.

Starter Problems: 3, 7, 8, 11, 15, 17, 25, 27, 35, 39.

Recommended Problems: 24, 33, 36, 40, 43, 46, 47, 51, 53, 57, 63, 69.

Lesson 4 – §2.4: Infinite Limits

A function $f(x)$ possesses an infinite limit at $x = a$ when its values grow larger and larger without bound as x approaches a. In this case one writes

$$\lim_{x\to a} f(x) = \infty.$$

As an example, $\lim_{x\to 0} 1/x^2 = \infty$.

In analogy, if the values of $f(x)$ are negative and grow larger and larger in magnitude as x approaches a, then the limit of $f(x)$ at $x = a$ is negative infinity, or

$$\lim_{x\to a} f(x) = -\infty.$$

One defines infinite limits for the one-sided limits $\lim_{x\to a^\pm} f(x)$ in a like fashion.

For infinite limits one has the following variant of the Squeeze Theorem: Suppose that $f(x) \geq g(x)$ for all x near, but not equal to, a, and $\lim_{x\to a} g(x) = \infty$, then also $\lim_{x\to a} f(x) = \infty$. As an exercise, formulate an analogous statement involving a negative infinity limit.

If at least one of the one-sided limits at $x = a$ is either ∞ or $-\infty$, the line $x = a$ is called a *vertical asymptote* of $f(x)$. If $f(x) = p(x)/q(x)$ is a rational function in reduced form (that is, $p(x)$, $q(x)$ share no common factors), then the vertical asymptotes are situated at the zeros of the denominator $q(x)$. The trigonometric functions $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$ possess an infinite number of vertical asymptotes. Can you locate all of them?

Starter Problems: 4, 5, 6, 8, 13, 17, 35, 38, 47.

Recommended Problems: 10, 15, 18, 26, 31, 39, 42, 44, 45, 51.

Lesson 5 – §2.5: Limits at Infinity

The limit

$$\lim_{x\to\infty} f(x)$$

of a function $f(x)$ as x approaches ∞ describes the values of $f(x)$ as x becomes larger and larger without bound. For example, since $1/x \approx 0$ when x is large, one can conclude that

$$\lim_{x\to\infty} \sin(\frac{1}{x}) = 0.$$

On the other hand,

$$\lim_{x\to\infty} (x^2 - x) = \infty,$$

since the term x^2 grows much faster than x. Finally,

$$\lim_{x\to\infty} \sin x$$

does not exist. Can you see why?

The limit $\lim_{x\to-\infty} f(x)$ of $f(x)$ at negative infinity is similarly found by letting x become negative and larger and larger in magnitude.

If either one of $\lim_{x\to\pm\infty} f(x) = L$ exists and is finite then the line $y = L$ is a *horizontal asymptote* for $f(x)$. Thus, as we saw above, the line $y = 0$ is a horizontal asymptote for $f(x) = \sin(1/x)$.

Suppose that $f(x) = p(x)/q(x)$ is a rational function with

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_o,$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_o,$$

where $a_m, b_n \neq 0$. Then

1. $\lim\limits_{x\to\pm\infty} \dfrac{p(x)}{q(x)} = 0$, if $m < n$,
2. $\lim\limits_{x\to\pm\infty} p(x)/q(x) = a_m/b_n$, if $m = n$,
3. $\lim\limits_{x\to\pm\infty} p(x)/q(x) = \pm\infty$, if $m > n$, depending on whether $m - n$ is even or odd.

Hence the rational function $f(x) = p(x)/q(x)$ has a horizontal asymptote precisely when the degree of $q(x)$ is equal or greater than the degree of $p(x)$.

One can use the exponent rules for the natural exponential function to conclude that

$$\lim_{x \to \infty} e^x = \infty, \qquad \lim_{x \to -\infty} e^x = 0.$$

Consequently, the logarithm function, as the inverse of the exponential function, must satisfy

$$\lim_{x \to 0} \ln x = -\infty, \qquad \lim_{x \to \infty} \ln x = \infty,$$

as you can also verify by graphing the functions.

Starter Problems: 4, 9, 11, 17, 19, 25, 45, 49, 52.

Recommended Problems: 12, 16, 27, 37, 39, 41, 47, 53, 63, 69, 71, 82.

Lesson 6 – §2.6: Continuity

A function $f(x)$ is *continuous* at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. That is, $f(x)$ is continuous at $x = a$, if

1. $f(x)$ is defined on some interval containing $x = a$,
2. the limit $\lim_{x\to a} f(x)$ exists, and
3. the limit is equal to the value $f(a)$.

Intuitively speaking, if $f(x)$ is continuous at $x = a$, then the graph of $f(x)$ contains no holes or gaps at $x = a$.

A function $f(x)$ is *continuous on an interval* if it is continuous at every point contained in that interval.

All algebraic functions and the basic transcendental functions $\sin x$, $\cos x$, e^x, $\ln x$ are continuous in their respective domains of definition.

The Limit Laws imply that the sum, the difference, the product, and the quotient of two continuous functions are again continuous in their domains of definition. Moreover, the composition of two continuous functions is continuous, as is the inverse function of a continuous function when it exists.

The Limit Laws also yield many other continuous functions. Can you see why the function

$$f(x) = \sqrt[4]{2 + \sin(\frac{1}{1+x^2})}$$

is continuous for all values of x?

The *Intermediate Value Theorem* can be used to locate solutions to equations. For example, you can apply the IVT to conclude that the seemingly complicated equation

$$x^7 - 5x^5 + x^2 + 1 = 0$$

must have a solution in the interval $(0, 1)$! You will find other applications of the IVT in the exercises below and in Laboratory Activity III.

Starter Problems: 4, 5, 9, 11, 13, 15, 21, 35, 41, 51, 59.

Recommended Problems: 17, 24, 28, 33, 37, 45, 49, 53, 55, 61, 74, 75, 85.

Lesson 7 – §3.1, 3.2: Introducing and Working with Derivatives

The *difference quotient* of a function $f(x)$ on the interval $[a, a+h]$ is given by

$$\frac{f(a+h)-f(a)}{h},$$

which can be interpreted as the slope of the secant line through the points $(a, f(a))$ and $(a+h, f(a+h))$, or, alternately, as the average rate of change of $f(x)$ on the interval $(a, a+h)$.

A function $f(x)$ is *differentiable* at the point $x = a$ if the limit of the difference quotient

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

exists (and is finite). One frequently also writes

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}.$$

(Check that the two definitions are equivalent!)

The value of the limit, $f'(a)$, is called the *derivative* of $f(x)$ at $x = a$. If $f(x)$ is differentiable at every point on an interval I, then the differentiation process determines a new function $f'(x)$, the derivative of $f(x)$, on I. One often employs the *Leibniz notation* $\frac{df}{dx}(x)$ to indicate the derivative $f'(x)$.

The computation of the derivative using the definition is illustrated in Examples 3, 4, and 5 of Section 3.1 and you should carefully go through the steps so that you will be able to carry out similar calculations on your own.

Basic applications of the derivative include the slope of the tangent line to the graph of a function and the instantaneous rate of change of a function. Thus, given the slope $m = f'(a)$ of the tangent line at $x = a$, its equation can be written as

$$y = m(x-a) + f(a).$$

In the same vein, if $x = s(t)$ is the position of a particle moving along the x-axis, its velocity is given by $\mathbf{v}(t) = \frac{ds}{dt}(t)$. (Note that the time t is now the independent variable, so the derivative must be computed with respect to t.)

If $f(x)$ is differentiable at $x = a$, then it is also continuous at $x = a$. Thus if $f(x)$ is *not* continuous at $x = a$, then it can not be differentiable at that point either. But keep in mind that there are many continuous functions that are not differentiable, as, for example, is the case with $f(x) = |x|$ at $x = 0$.

Finally, note that the characterization of a non-differentiable function given on p. 140 in the text is far from comprehensive. For example, the function $f(x) = \sqrt{|x|}\cos x$ satisfies none of the conditions a, b, or c, but it is still not differentiable at $x = 0$ (show it!).

Starter Problem List:

Section 3.1: 3, 9, 15, 28, 41.

Section 3.2: 3, 5, 8, 15, 30.

Recommended Problem List:

Section 3.1: 6, 13, 23, 33, 51, 53, 55.

Section 3.2: 9, 11, 13, 20, 22, 26, 33 a.

Lesson 8 – §3.3: Rules of Differentiation

The following rules for the derivative can be derived directly from the definition.

1. **Constant function rule:** $\frac{d}{dx}c = 0.$
2. **Power rule:** $\frac{d}{dx}x^n = nx^{n-1}, \qquad n = 1, 2, 3, \ldots.$
3. **Constant multiple rule:** $\frac{d}{dx}\big[cf(x)\big] = cf'(x).$
4. **Sum rule:** $\frac{d}{dx}\big[f(x) + g(x)\big] = f'(x) + g'(x).$

As you will see in Lesson 14, the power rule $\frac{d}{dx}x^r = rx^{r-1}$, in fact, holds true for any real exponent r.

One often needs to combine the above rules to compute the derivative of a given function. For example, by applying both the constant multiple and sum rules we see that

$$\frac{d}{dx}(2f(x) - 5g(x)) = 2f'(x) - 5g'(x).$$

Notice that in order to compute a derivative at a particular point using any of the four rules above, *you only need to know the value of the derivatives of the constituent functions at that point*. So if we know that $f'(1) = -1$, $g'(1) = 3$, then

$$\frac{d}{dx}(2f(x) - 5g(x))|_{x=1} = 2f'(1) - 5g'(1) = -17.$$

Euler's number $e = 2.718\ldots$ is defined by the property that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

As a consequence,

$$\frac{d}{dx}e^x = e^x,$$

that is, the natural exponential function is its own derivative!

The higher order derivatives

$$\frac{d^2}{dx^2}f(x) = f''(x), \quad \frac{d^3}{dx^3}f(x) = f'''(x), \quad \ldots$$

of a function $f(x)$ are obtained by repeatedly differentiating $f(x)$. For example,

$$\frac{d^2}{dx^2}x^7 = \frac{d}{dx}\big(\frac{d}{dx}x^7\big) = 7\frac{d}{dx}x^6 = 42x^5.$$

As a exercise, compute $\frac{d^4}{dx^4}x^3$. What do you notice? Can you generalize your observation into a rule for higher order derivatives of a polynomial function?

Starter Problems: 3, 9, 15, 17, 27, 29, 35, 37.

Recommended Problems: 10, 16, 21, 26, 33, 39, 41, 46, 55, 58, 60, 72.

Lesson 9 – §3.4: The Product and Quotient Rules

In this section you will learn rules for computing the derivatives of the product and quotient of two functions in terms of the values of the functions and their derivatives.

1. Product rule: $\dfrac{d}{dx}\big[f(x)g(x)\big] = f'(x)g(x) + f(x)g'(x).$

2. Quotient rule: $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\,.$

Importantly, these rules allow one to compute the derivative of the product and quotient of two functions at a point from the values of the functions and their derivatives at that point only! As an example, suppose that the following information is given:

$$f(-1) = 3, \quad f'(-1) = -1, \quad g(-1) = 2, \quad g'(-1) = -3.$$

Then by the quotient rule,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]\Big|_{x=-1} = \frac{f'(-1)g(-1) - f(-1)g'(-1)}{g(-1)^2} = \frac{7}{4}\,.$$

In more complicated derivative computations you will need to be able to combine these (and other) rules of differentiation; see Example 7 in the text for a typical application.

The quotient rule can also be used to extend the power rule of differentiation from Lesson 8 to negative values of the exponent.

The chain rule of Lesson 12 will furnish an alternate form of the quotient rule useful in convoluted examples.

The horizontal scaling $f(x) \to f(kx)$ changes the instantaneous rate of change of the function by the factor k. Thus $\dfrac{d}{dx}f(kx) = kf'(kx)$, so, in particular,

$$\frac{d}{dx}\,e^{kx} = ke^{kx}, \qquad \text{for any real number } k.$$

The function e^{kx} appears in models for population growth and radioactive decay.

Starter Problems: 6, 8, 15, 19, 33, 37, 43, 62, 74.

Recommended Problems: 10, 18, 21, 25, 35, 39, 47, 51, 55, 60, 65, 69, 79, 81.

Lesson 10 – §3.5: Derivatives of Trigonometric Functions

The following limits, which are important on their own right, are used to compute the derivatives of the sine and cosine functions:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

Both expressions are in indeterminate form, so finding the required limits by a geometric argument, as is done in the text, takes some work. It is important to bear in mind that in the two limit statements the x variable must be expressed in radians.

You should commit to memory the derivative formulas of the basic trigonometric functions cataloged in the table below.

1. $\frac{d}{dx} \sin x = \cos x$	**2.** $\frac{d}{dx} \cos x = -\sin x$
3. $\frac{d}{dx} \tan x = \sec^2 x = (\cos x)^{-2}$	**4.** $\frac{d}{dx} \cot x = -\csc^2 x = -(\sin x)^{-2}$
5. $\frac{d}{dx} \sec x = \sec x \tan x = \frac{\sin x}{\cos^2 x}$	**6.** $\frac{d}{dx} \csc x = -\csc x \cot x = -\frac{\cos x}{\sin^2 x}$

Note the antisymmetry in the formulas for the derivatives of the sine and cosine functions. The remaining derivatives 3-6 can be obtained from these two with the help of the quotient rule, as you should verify by yourself. Typically the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are given in terms of the same, but keep in mind that when manipulating or simplifying a complicated trigonometric expression, it is often helpful to convert it first to a form involving the sine and cosine functions only.

Caveat: In all the derivative formulas for trigonometric functions x must be expressed in radians. If you use other units (degrees, grads, ...), you will need to modify the formulas accordingly. Thus, for example, if z is given in degrees, then

$$\frac{d}{dz} \sin z = \frac{\pi}{180} \cos z,$$

as you should confirm on your own.

Starter Problems: 4, 7, 11, 17, 21, 34.

Recommended Problems: 9, 15, 23, 26, 33, 38, 43, 51, 55.

Lesson 11 – §3.6: Derivatives as Rates of Change

The key concepts in this lesson are the *average rate of change* and the *instantaneous rate of change* of a quantity that varies in time t.

Suppose that an object moves along the x-axis so that its position is given by $x = f(t)$, where t denotes the time. For example, $x = \cos t$ describes an instance of *harmonic motion*. Then the displacement of the object during a time period from $t = a$ to $t = a + \Delta t$ is $\Delta x = f(a + \Delta t) - f(a)$, so the *average velocity* of the object is given by

$$\mathbf{v}_{\text{ave}} = \frac{\Delta x}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}, \tag{11.1}$$

where, as you notice, the right-hand side is simply the difference quotient of the function $f(t)$ on the interval $[a, a + \Delta t]$.

As you can easily check, the average velocity in harmonic motion $x = \cos t$ over the interval $[0, 2\pi]$ is zero even though the object moves back and forth between 1 and -1. Thus the average velocity does not often accurately characterize motion. A more precise description is obtained by computing the average velocity on increasingly smaller intervals, that is, by letting $\Delta t \to 0$ in (11.1). But the limit (once again) results in the derivative and so the *(instantaneous) velocity* of the particle is simply

$$\mathbf{v} = f'(t).$$

The instantaneous rate of change of velocity is the *acceleration*, so

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t).$$

(Warning: Don't be confused by the text's use of the same symbol a for a fixed value of the x-coordinate and the acceleration of an object.)

Example 5 in the text deals with a business application. Suppose that the cost of producing x items in a manufacturing plant can be computed from the *cost function* $C(x)$. Then the *average cost* of producing x items is $C(x)/x$, while the *marginal cost* measures the expense of manufacturing one additional item. The increase in production from x to $x + \Delta x$ items incurs the extra average expense of

$$\frac{C(x + \Delta x) - C(x)}{\Delta x}$$

per item. The marginal cost is obtained by taking the limit as $\Delta x \to 0$, and so is given by the derivative $C'(x)$ of the cost function. Of course, in real life situation, x

and Δx take on integral values, so the marginal cost will only give an approximation to the actual increase in the cost of producing one more item,

$$C(x+1) - C(x) \approx C'(x).$$

Starter Problems: 6, 7, 9, 12, 17, 21, 25, 38.

Recommended Problems: 10, 15, 19, 24, 26, 31, 37, 41, 48, 51.

Lesson 12 – §3.7: The Chain Rule

The *chain rule* allows you to compute the derivative of the composition of two functions $f(g(x))$ in terms of the derivatives $f'(x)$ and $g'(x)$.

Chain Rule: $$\frac{d}{dx}\big[f(g(x))\big] = f'(g(x))g'(x)$$

You will see many variants of this basic formula such as

$$\frac{d}{dt}f(h(t)) = f'(h(t))h'(t), \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \qquad \text{and} \qquad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

but they all express the same fact as the boxed formula, only in different notation.

The chain rule is one of the most useful tools for finding the derivatives of complicated functions as it lets you break down the computation into the differentiation of simpler functions.

Perhaps the following argument will help you to understand and remember the chain rule. Recall that if x changes by a small amount Δx, then the change in the values of the function $g(x)$ is approximately

$$g(x + \Delta x) - g(x) \approx g'(x)\Delta x.$$

Thus, in the composition $f(g(x))$, the argument (or the input) of $f(y)$ changes by $\Delta y \approx g'(x)\Delta x$, where we have written $y = g(x)$. So the value of $f(g(x))$ changes by

$$f(y + \Delta y) - f(y) \approx f'(y)\Delta y \approx f'(g(x))g'(x)\Delta x.$$

But this must be equal to $\frac{d}{dx}\big[f(g(x))\big]\Delta x$, so by comparing the two expressions for the change of $f(g(x))$, you will recover the chain rule $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$.

Example. If $f(y) = y^n$ with the derivative $f'(y) = ny^{n-1}$, then the chain rule reduces to

$$\frac{d}{dx}\big[g(x)^n\big] = ng(x)^{n-1}g'(x).$$

For $n = -1$, this, combined with the product rule, yields an alternate useful form

$$\frac{d}{dx}\left[\frac{h(x)}{g(x)}\right] = \frac{d}{dx}\big[h(x)g(x)^{-1}\big] = h'(x)g(x)^{-1} - h(x)g(x)^{-2}g'(x)$$

of the quotient rule. □

Example. A derivative computation often requires a repeated application of the chain rule. For example,

$$\begin{aligned}\frac{d}{dx}\sin(\cos(x^2)) &= \cos(\cos(x^2))\frac{d}{dx}\cos(x^2) = \cos(\cos(x^2))(-\sin(x^2))\frac{d}{dx}(x^2)\\ &= \cos(\cos(x^2))(-\sin(x^2))2x = -2x\cos(\cos(x^2))\sin(x^2),\end{aligned}$$

where, at first, we used the chain rule with $f(y) = \sin(y)$, $g(x) = \cos(x^2)$ and, in computing the derivative of $\cos(x^2)$, with $f(y) = \cos(y)$, $g(x) = x^2$. Example 6 in the text highlights the same type of computation. □

Example. In light of the chain rule, the rate of change of the composition $f(g(x))$ at $x = a$ is the product of the rates of change of $f(y)$ at $y = g(a)$ and of $g(x)$ at $x = a$. As an illustration, suppose we know that $g(0) = 1$, $g'(0) = 3$, $f'(1) = -1$. Then

$$\frac{d}{dx}\left[f(g(x))\right] |_{x=0} = f'(g(0))g'(0) = f'(1)3 = -3.$$

Thus the rate of change of $f(g(x))$ at $x = 0$ is -3. □

Starter Problems: 2, 6, 7, 11, 16, 21, 24, 44.

Recommended Problems: 15, 20, 29, 37, 41, 45, 52, 62, 66, 70, 77, 90.

Lesson 13 – §3.8: Implicit Differentiation

Suppose your task is to compute the slope of the tangent line to the ellipse $4x^2+y^2 = 1$ at the point $(\frac{1}{4}, \frac{\sqrt{3}}{2})$. You can, of course, solve the equation for y to find that

$$y = \pm\sqrt{1 - 4x^2},$$

which can then be differentiated to obtain the slope.

But as an alternate approach, note that the equation for the ellipse defines (up to a $\pm$ sign) y as a function of x. Write $y = f(x)$ for the function so that

$$4x^2 + f(x)^2 = 1.$$

But this equation simply states that the function $4x^2 + f(x)^2$ on the left-hand side must be identically constant 1! Hence its derivative must vanish, so, by the chain rule for powers, we obtain

$$8x + 2f(x)f'(x) = 0, \qquad \Longrightarrow \qquad f'(x) = -\frac{4x}{f(x)}.$$

Thus the slope at $(\frac{1}{4}, \frac{\sqrt{3}}{2})$ will be $f'(\frac{1}{4}) = -\frac{2}{\sqrt{3}}$.

One typically forgoes writing $f(x)$ for y and directly differentiates both sides of the defining equation $4x^2 + y^2 = 1$ while keeping in mind that y is a function of x. This gives

$$8x + 2yy' = 0 \quad \Longrightarrow \quad y' = -\frac{4x}{y}.$$

It is now a simple matter to find the slope by plugging in the x and y-coordinates of the base point.

This process of differentiating y without knowing its explicit expression in terms of x is called *implicit differentiation*. As we saw, it simplified the task of finding the slope of the tangent line to the ellipse, but implicit differentiation really becomes an indispensable tool in problems in which one can not analytically solve y in terms of x. (It can be shown that the equation $y + e^{x^2 y} = 1$ defines y as a function of x, but can you find an explicit expression for y in terms of x? How about computing dy/dx in terms of x and y?)

Caveat: Common mistakes in carrying out implicit differentiation are to forget to apply the chain rule when differentiating terms involving y (which is to be considered a function of x) and to forget to differentiate constant terms.

Two lines ℓ_1, ℓ_2 intersects perpendicularly (or at 90° angle) if their slopes satisfy $s_1 s_2 = -1$, provided that $s_1 \neq 0, \infty$. You can use this property to find the *normal line* to a plane curve at a point: since the slope of the tangent line is dy/dx, the slope of the normal line will be $-1/(dy/dx)$ (The normal line is vertical when $dy/dx = 0$).

Example. Example 6 in the text asks for the slope of the tangent line to the curve $2(x+y)^{1/3} = y$ at $(4,4)$. As in the text, differentiate the equation implicitly to get

$$\frac{2}{3}(x+y)^{-2/3}\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}\,.$$

Since the goal is to find the slope at a given point, there is no need to solve the equation to derive a general formula for dy/dx – one can instead directly substitute the coordinates of the base point to see that

$$\frac{2}{3}(4+4)^{-2/3}\left(1+\frac{dy}{dx}\right) = \frac{1}{6}\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}\,,$$

from which one finds that $dy/dx = 1/5$. Hence, in particular, the tangent and normal lines to the curve at $(4,4)$ are given by

$$y = \frac{x}{5} + \frac{16}{5}, \qquad y = -5x + 24,$$

respectively. You can exploit the same shortcut to facilitate solving several of the exercises in this section. □

Starter Problems: 1, 3, 7, 11, 15, 25, 32, 37, 65.

Recommended Problems: 10, 17, 23, 30, 35, 39, 41, 49, 53, 57, 61, 70.

Lesson 14 – §3.9: Derivatives of Logarithmic and Exponential Functions

You recall that the natural exponential function is one-to-one, so it has an inverse, which is called the *natural logarithmic function* $\ln x$. The domain of $\ln x$ is $(0, \infty)$ and the range is $(-\infty, \infty)$, as you should verify by graphing $\ln x$ using the graph of e^x as the starting point. Thus, by definition,

$$e^{\ln x} = x, \qquad \text{for } x > 0, \qquad \text{and} \qquad \ln e^x = x, \qquad \text{for all } x.$$

The power rules for the exponential function have their counterparts, the *logarithm rules*, which are summarized below.

1. $e^x e^y = e^{x+y} \iff \ln(xy) = \ln x + \ln y.$

2. $\left(e^x\right)^y = e^{xy} \iff \ln(x^y) = y \ln x.$

These formulas also yield the expression for the derivative of the natural logarithm function: Differentiate the equation $x = e^{\ln x}$ with respect to x (and don't forget to apply the chain rule) to see that

$$1 = e^{\ln x} \frac{d}{dx} \ln x = x \frac{d}{dx} \ln x,$$

where, in the second step, we used the relation $x = e^{\ln x}$. The resulting equation can be easily solved:

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}\,.}$$

As we will see later, the above computation is just a special case of a general algorithm for determining the derivative of an inverse function.

The *logarithm function in base* b, $b > 0$, is defined by the relations

$$y = \log_b x \iff x = b^y, \qquad \text{where } x > 0.$$

You should derive the identities

$$b^x = e^{x \ln b}, \qquad \log_b x = \frac{\ln x}{\ln b}\,,$$

which, by differentiation, give

$$\frac{d}{dx}b^x = (\ln b)b^x, \qquad \frac{d}{dx}\log_b x = \frac{1}{x\ln b}\,,$$

for the derivatives of the exponential and logarithm function in base b.

Logarithmic differentiation is founded on the observation that

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)} \qquad \Longleftrightarrow \qquad f'(x) = f(x)\frac{d}{dx}\ln f(x).$$

Thus one can find the derivative of a function $f(x)$ by starting with differentiating $\ln f(x)$ and then multiplying the result by $f(x)$. For certain type of functions this will simplify computing the derivative. Can you think of any examples?

Example. As an application of logarithmic differentiation we compute $\frac{d}{dx}(\sin x)^x$. We have that

$$\begin{aligned}\frac{d}{dx}(\sin x)^x &= (\sin x)^x\frac{d}{dx}\ln(\sin x)^x = (\sin x)^x\frac{d}{dx}(x\ln\sin x)\\ &= (\sin x)^x\big(\ln\sin x + x\frac{\cos x}{\sin x}\big) = (\sin x)^x\ln\sin x + x\cos x(\sin x)^{x-1}.\end{aligned}$$

□

Starter Problems: 3, 7, 11, 23, 34, 56, 62, 77.

Recommended Problems: 13, 15, 21, 26, 33, 37, 47, 51, 55, 61, 67, 73, 81, 89.

Lesson 15 – §3.10: Derivatives of Inverse Trigonometric Functions

Suppose that $f(x)$ is invertible with the inverse function $f^{-1}(x)$. Then, by the very definition, $f(x)$ and $f^{-1}(x)$ satisfy $f(f^{-1}(x)) = x$. We can differentiate this identity to see that

$$f'(f^{-1}(x))\frac{d}{dx}f^{-1}(x) = 1.$$

(Can you see why we again needed to use the chain rule?) Solving for the derivative of the inverse function, we obtain

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}\,.$$

In other words, the *derivative of* $f^{-1}(x)$ *is the reciprocal of* $f'(x)$ *composed with* $f^{-1}(x)$!

Example. In this example we compute the value of the derivative of $f^{-1}(x)$ at $x = 3$ assuming that $f(x)$ is invertible and that we are given the following information:

$$f(1) = 3, \quad f(3) = -2, \quad f'(1) = 6, \quad f'(3) = 4.$$

In order to find $\frac{d}{dx}f^{-1}(x)|_{x=3}$ we apply the basic identity for the derivative of the inverse function. For this, we will need $f^{-1}(3)$, which, since $f(1) = 3$, equals 1. Thus

$$\frac{d}{dx}f^{-1}(x)|_{x=3} = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)} = \frac{1}{6}\,.$$

Note that in order to compute $\frac{d}{dx}f^{-1}(x)|_{x=1}$, we would need need to know the value of $f^{-1}(1)$, that is, to find an x value such that $f(x) = 1$. But this is not given in the statement of the problem, so the information provided does not allow us to compute the derivative of the inverse function at $x = 1$. □

The sine function is 2π-periodic, so it fails the horizontal line test and can not be invertible. However, its *principal part*, the restriction of $\sin x$ to the interval $[-\pi/2, \pi/2]$, is one-to-one and possesses an inverse function, which is denoted by $\sin^{-1} x$ (or often also by $\arcsin x$). Thus

$$y = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \qquad \Longleftrightarrow \qquad x = \sin^{-1} y, \quad -1 \le x \le 1.$$

As an exercise, graph $\sin^{-1} x$ starting from the graph of $\sin x$, carefully identifying the domain and range of $\sin^{-1} x$ in your plot.

You should note that the inverse function of $\sin x$ can be defined on any interval, for example, on $[-3\pi/2, -\pi/2]$, $[\pi/2, 3\pi/2]$, $\ldots$, where it is one-to-one. The choice of the particular interval is usually dictated by the problem at hand.

Caveat 1. On account of the definitions,

$$\sin(\sin^{-1}(x)) = x, \quad \text{for all } -1 \leq x \leq 1,$$

but

$$\sin^{-1}(\sin x) = x, \quad \textit{only when} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\,.$$

Can you derive an expression for $\sin^{-1}(\sin x)$ that holds for any x?

Caveat 2. Be sure to distinguish between the notation used for the inverse sine and for the reciprocal of the sine function:

$$\sin^{-1} x = \arcsin x, \qquad (\sin x)^{-1} = \frac{1}{\sin x}\,.$$

In order to find the derivative of $\sin^{-1}(x)$, we apply the general procedure for differentiating inverse functions. This gives

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))}\,.$$

We still need to simplify the composition $\cos(\sin^{-1}(x))$, which will be based on the basic trigonometric identity $\cos^2\theta + \sin^2\theta = 1$. If you substitute $\theta = \sin^{-1} x$ into this and use the identity $\sin(\sin^{-1}(x)) = x$, you will see that

$$\cos^2(\sin^{-1} x) + x^2 = 1.$$

When solving for the cosine term, keep in mind that the values of $\theta = \sin^{-1}(x)$ fall within the interval $[-\pi/2, \pi/2]$, where $\cos\theta$ is *positive*, so the above equation yields

$$\cos(\sin^{-1} x) = \sqrt{1 - x^2}\,.$$

In conclusion,

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}\,.$$

The inverse functions of the other basic trigonometric functions $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are similarly defined by a restriction to a suitable interval, see Section 1.4 in the text. The derivatives of the inverse functions are then computed using the general process as explained above; see the table below.

$f(x)$	*Domain and Range of Principal Part*		$\frac{d}{dx}f^{-1}(x)$
1. $\sin x$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$,	$[-1, 1]$	$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
2. $\cos x$	$[0, \pi]$,	$[-1, 1]$	$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
3. $\tan x$	$(-\frac{\pi}{2}, \frac{\pi}{2})$,	$(-\infty, \infty)$	$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
4. $\cot x$	$(0, \pi)$,	$(-\infty, \infty)$	$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$
5. $\sec x$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$,	$(-\infty, -1] \cup [1, \infty)$	$\frac{d}{dx}\sec^{-1} x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
6. $\csc x$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$,	$(-\infty, -1] \cup [1, \infty)$	$\frac{d}{dx}\csc^{-1} x = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

Starter Problems: 3, 5, 7, 13, 21, 32, 37, 47, 61.

Recommended Problems: 9, 15, 25, 29, 35, 42, 45, 49, 54, 67.

Lesson 16 – §3.11: Related Rates

Suppose that two quantities, changing with time t, are related by an equation. Then, if we know the rate of change of one of the quantities, we can find the rate of change of the other one by differentiating the relating equation with respect to t. As a specific example, consider the x and y coordinates of an object moving along the ellipse $4x^2 + y^2 = 1$. Then x and y are functions of time, so by computing the t derivative of the equation of the ellipse (and again applying the chain rule), we see that

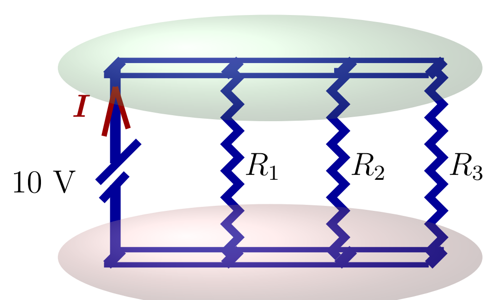
$$8x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad \Longleftrightarrow \qquad \frac{dy}{dt} = -\frac{4x}{y}\frac{dx}{dt}\,.$$

Thus, if we know the position of the object and the rate of change of the x coordinate, we can readily compute the rate of change of the y coordinate.

This simple example underscores the general fact that related rate problems typically reduce to differentiating an equation involving two or more quantities changing in time. The differentiation step entails applying the chain rule very much like in problems requiring implicit differentiation. Many of the exercises in the text involve basic geometry and call for a bit of creativity, but steps for solving related rates problems outlined in the section will help you to get on the right track. A typical example involving related rates is presented below.

Example. A circuit consists of three resistors with resistance R_1, R_2, and R_3 set in parallel and a 10 volt power source. The value of R_1 is increasing at the rate of $0.4\ \Omega/s$ and that of R_2 is decreasing at the rate of $0.7\ \Omega/s$, while R_3 stays constant. Find the rates of change of the total resistance and the current in the circuit when $R_1 = 20\ \Omega$, $R_2 = 35\ \Omega$, and $R_3 = 5\ \Omega$.

Solution: First draw a picture to ensure you comprehend the physical set-up and understand the question.



You may recall from your physics courses that the total resistance R of three resistors set in parallel is determined by the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}, \qquad (16.1)$$

and that the total current across the three resistors is given by $I = V/R$. These two equations express the basic relationships between the variables in the problem.

We will start with finding the rate of change of the current as this is somewhat easier computation of the two to carry out.

Since voltage V is assumed to be constant, the equation for current, after differentiating with respect to time t and applying the chain rule, yields

$$\frac{dI}{dt} = -\frac{V}{R^2}\frac{dR}{dt}\,. \tag{16.2}$$

Hence we can determine the rate of change of the current I once we know the value of $R^{-2}dR/dt$. But this quantity can be derived by differentiating equation (16.1) for the total resistance. We have

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt} - \frac{1}{R_3^2}\frac{dR_3}{dt}\,. \tag{16.3}$$

We will omit units in the subsequent computations. After substituting the given values

$$R_1 = 20, \quad R_2 = 35, \quad R_3 = 5, \quad \frac{dR_1}{dt} = 0.4, \quad \frac{dR_2}{dt} = -0.7, \quad \frac{dR_3}{dt} = 0, \tag{16.4}$$

into (16.3), we find that

$$\frac{1}{R^2}\frac{dR}{dt} = \frac{0.4}{20^2} + \frac{-0.7}{35^2} + 0 = \frac{3}{7000} \approx 4.3 \times 10^{-4}.$$

Consequently, by equation (16.2),

$$\frac{dI}{dt} = -\frac{3}{700} \approx -4.3 \times 10^{-3}\, A/s.$$

Next, by equation (16.1) for the total resistance,

$$\frac{1}{R} = \frac{39}{140} \qquad \Longrightarrow \qquad R = \frac{140}{39}\,,$$

whence, by (16.3),

$$\frac{dR}{dt} \approx 5.5 \times 10^{-3}\, \Omega/s.$$

□

Example. Example 4 in the text deals with the rate of change of the angle $\theta(t)$ between the horizontal and an observer's line of sight to a rising balloon. This problem can be disposed of in short order by applying the ideas you learned in Lesson 13.

From the picture in the text,

$$\tan(\theta(t)) = \frac{y(t)}{200},$$

which can be directly differentiated to yield

$$\sec^2(\theta(t))\frac{d\theta}{dt}(t) = \big(1+\tan^2(\theta(t))\big)\frac{d\theta}{dt}(t) = \frac{1}{200}\frac{dy}{dt}(t). \tag{16.5}$$

By the statement of the problem, $\frac{dy}{dt}(t) = 4$, so $y(30) = 120$ and $\tan(\theta(30)) = 3/5$. Now (16.5) can be solved for the rate of change of θ to give

$$\frac{d\theta}{dt}(30) = \frac{4}{200(1+(3/5)^2)} = \frac{1}{68} \approx 0.0147 \text{ (rad/s)}.$$

□

Starter Problems: 2, 5, 9, 12, 18, 28, 34.

Recommended Problems: 7, 16, 19, 23, 29, 33, 35, 37, 45.

Lesson 17 – §4.1: Maxima and Minima

Many applications of calculus to concrete problems call for finding the maximum or the minimum value of a pertinent function, which, for example, could be the revenue or cost function, the acceleration of an object, air drag, the size of a population, and so on.

A value $f(c)$ is a *local maximum* of a function $f(x)$ if there is some interval $(c-h, c+h)$, $h > 0$, such that $f(c)$ is the largest value of $f(x)$ in this interval (so $f(c) \geq f(x)$ for any x between $c - h$ and $c + h$, but there can be x values *outside the interval* with $f(x) > f(c)$). One similarly defines a *local minimum*.

A function assumes an *absolute maximum* value at $x = c$ if $f(c) \geq f(x)$ for any x in the domain of the function. Again, the *absolute minimum* of $f(x)$ is defined similarly.

Note that a function needs not possess any local or absolute maxima or minima as you can see, for example, by graphing the function $f(x) = x^3$ or $f(x) = \tan x$. However, a continuous function defined on a *finite and closed* interval always attains an absolute maximum and minimum on that interval, which could also be realized at the endpoints of the interval. This fact is known as the *extreme value theorem*, and, as is easily verified by the way of examples, continuity is a necessarily condition for the theorem to hold.

If $f(c)$ yields a local maximum for a differentiable function $f(x)$, then the slope of the graph of $f(x)$ must be non-negative immediately to the left of c and non-positive immediately to the right of c. Thus, at $x = c$, the derivative of $f(x)$ must vanish, that is, $f'(c) = 0$. An analogous statement holds for local minima. Hence, in general, *if $f(c)$ is a local maximum or minimum of $f(x)$, then either $f'(c)$ does not exist or $f'(c) = 0$!* Points contained in the domain of $f(x)$ that satisfy either one of these two conditions are collectively called *critical points* of $f(x)$ and they can often be effectively used to locate local maxima and minima, and, with the help of these, also the absolute maximum and minimum of a given function.

Starter Problems: 5, 11, 15, 23, 37, 41, 56.

Recommended Problems: 21, 25, 35, 43, 46, 53, 61, 67, 69, 75, 77.

Lesson 18 – §4.2: What Derivatives Tell Us

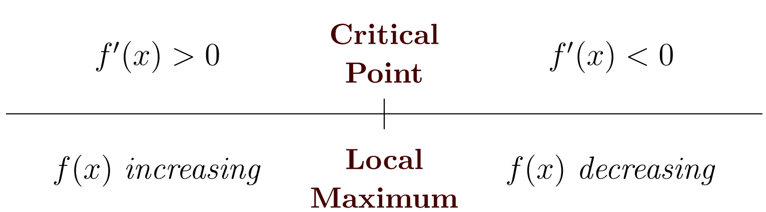
The first and second derivatives are intimately related to the behavior of a function and they can be effectively used in plotting graphs. Roughly speaking, the derivative $f'(x)$ measures the direction of the graph, or the steepness of the graph, and the second derivative measures the rate at which this direction is changing, so, intuitively speaking, $f''(x)$ yields a measurement for the *curvature* of the graph.

A function $f(x)$ is *(strictly) increasing* on an interval I if $x_2 > x_1$ implies that $f(x_2) > f(x_1)$ for all x_1, x_2 on I, and *(strictly) decreasing* if $x_2 > x_1$ implies that $f(x_2) < f(x_1)$ for all x_1, x_2 on I. One can detect intervals on which $f(x)$ is increasing or decreasing by the sign of the derivative:

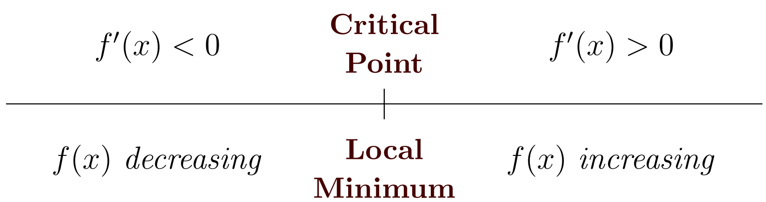
$$\begin{array}{lll} f'(x) > 0 \quad \text{for all } x \text{ on } I & \Longrightarrow & f(x) \text{ is increasing on } I. \\ f'(x) < 0 \quad \text{for all } x \text{ on } I & \Longrightarrow & f(x) \text{ is decreasing on } I. \end{array}$$

Intuitively, $f'(x) > 0$ implies that the slope of the tangent line is pointing up so the graph of the function is tending higher when x increases, that is, $f(x)$ is increasing. Analogously, $f'(x) < 0$ implies that the graph is tending lower so the function must be decreasing. But note that the converse of the above derivative test does not hold: a function increasing on an interval I can have $f'(x) = 0$ at some (in fact, even at infinitely many) x on I.

Next assume that c is a critical point of $f(x)$ and that $f'(x) > 0$ on some interval (a, c) left of c and $f'(x) < 0$ on some interval (c, b) to the right of c. Then as x approaches c from the left, $f(x)$ increases and, as x moves past c, it decreases. Hence $f(x)$ must have a local maximum at c; see the diagram below.

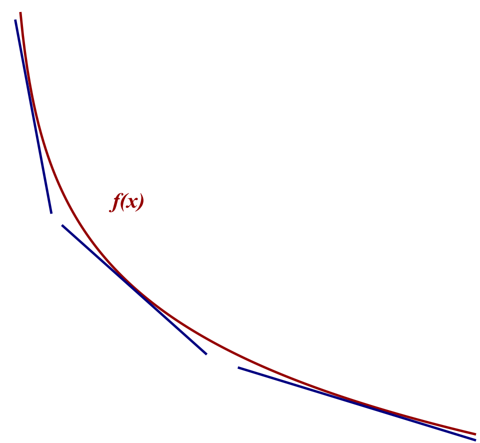


Similarly, $f'(x) < 0$ on (a, c) and $f'(x) > 0$ (c, b) implies that $f(c)$ must be a local minimum for $f(x)$.



Finally, if the derivative $f'(x)$ does not change sign at $x = c$, $f(c)$ will not be a local maximum or minimum.

The above arguments can also be applied to the function $f'(x)$ (provided, of course, that its derivative $f''(x)$ exists). Thus if $f''(x) > 0$ on an interval I, the derivative $f'(x)$ is increasing. In this situation the tangent lines drawn along the graph of $f(x)$ turn counterclockwise when x increases and the function $f(x)$ is said to be *concave up* on I.
If, in turn, $f''(x) < 0$ on an interval I, then the tangent lines turn clockwise along the graph of the function, and $f(x)$ is said to be *concave down* on I.



Graph of a function concave up

A point c in the domain of $f(x)$ is called an *inflection* point for $f(x)$ if concavity changes at c. In particular, you can locate inflection points by finding all points c at which $f''(c)$ does not exist or at which $f''(c) = 0$. If $f''(x)$ changes sign at c, then c will be an inflection point. But bear in mind that a point c in the domain of $f(x)$ can be an inflection point also when $f''(c)$ does not exist.

These type of arguments also yields the *second derivative test for local extrema*: Suppose that $f''(x)$ exists on an interval I and $x = c$ is an critical point on I with $f'(c) = 0$. If $f''(c) > 0$, then $f(c)$ is a local *minimum* for $f(x)$ while if $f''(c) < 0$, then $f(c)$ is a local maximum for $f(x)$.

The fact that positive second derivative is associated with a local minimum might seem counter-intuitive at first, but the rule is easy to remember in its correct form by considering the parabola $f(x) = x^2$ for which $f''(0) > 0$ at the local (in fact, absolute) minimum at $x = 0$.

The second derivative test is inconclusive if $f'(c) = f''(c) = 0$, and in this case you should investigate the sign of the derivative $f'(x)$ on each side of c.

Starter Problems: 6, 8, 11, 17, 27, 31, 40, 53, 57, 73.

Recommended Problems: 13, 23, 26, 33, 41, 48, 51, 55, 61, 81, 84, 87, 103.

Lesson 19 – §4.3: Graphing Functions

As we saw in the previous lesson, the first and second derivatives are closely associated with the properties of a function. It then comes as no surprise that they also yield a wealth of information that is valuable in analyzing the graph of the function.

Caveat: When plotting a function, do not rely on your graphing utility to do the work for you but perform the steps outlined below on your own and by hand. Only when you are done with the problem should you double-check your results against the graph produced by software.

In a typical graphing problem you should follow the steps outlined below.

1. Identify the interval on which the function $f(x)$ is to be graphed. This can be a subset of the domain of $f(x)$.

2. Check for special properties of $f(x)$. Is the function even ($f(-x) = f(x)$) or odd ($f(-x) = -f(x)$), or, more generally, symmetric with respect to the line $x = c$ ($f(2c - x) = f(x)$) or antisymmetric with respect to it ($f(2c - x) = -f(x)$)? Is it periodic ($f(x + L) = f(x)$) or antiperiodic ($f(x + L) = -f(x)$)? (How would each of these properties help you with drawing the graph of a function?)

3. Find the intercepts of the graph with the x and y-axes.

4. Compute the derivative $f'(x)$ and locate all critical points. Identify intervals on which $f(x)$ is increasing or decreasing, and classify all local and absolute extrema.

5. Compute the second derivative $f''(x)$ and locate all inflection points. Identify intervals on which $f(x)$ is concave up or down. You may also double-check your classification of local extrema in step 3 by the second derivative test.

6. Compute the limits $\lim_{x\to\pm\infty} f(x)$ and locate vertical and horizontal asymptotes, if any.

7. Graph the function with the help of all the information you have gathered in the above steps. And keep in mind that there is no substitute for doing it yourself. Practice makes perfect!

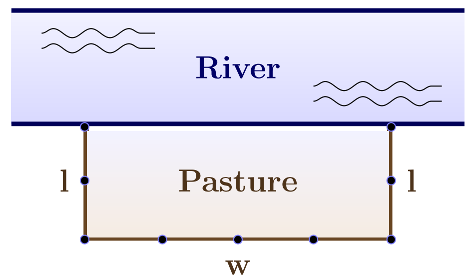
Starter Problems: 2, 6, 7, 9, 18, 23, 29, 37, 44.

Recommended Problems: 8, 11, 17, 29, 33, 39, 42, 45, 49, 54, 63, 64.

Lesson 20 – §4.4: Optimization Problems

Finding the best possible solution in practical problems is one of the principal applications of calculus. Identifying these *optimal* solutions is based on the min/max problems of Lesson 17, but, besides calculus, some experience with general problem solving will be called for. The following example illustrates the typical procedure.

Example. A farmer with 1200 m of fencing wants to enclose a rectangular pasture that borders a straight river. He needs no fence along the river. Find the largest possible area the pasture.



1. The problem asks for the maximum area of a rectangular field, so the relevant quantities in this problem are the width w along the river, the length l, and the area A of the pasture.

2. The quantity to be maximized is the area which is given in terms of the other quantities by $A = wl$. This is a function of two variables and we have not learned yet how to find the extreme values of such functions.

3. However, the problem imposes a constraint between w and l: The total length of the three sides of the pasture is 1200, so $w + 2l = 1200$. (Recall that we are assuming that the river runs along the width of the pasture. We will also omit units from now on.)

4. The constraints between w and l can be solved, say, for w to yield $w = 1200 - 2l$. This expression, when substituted into the expression for A, gives $A = (1200 - 2l)l$, a function of one variable. Both l and w are be positive, so l must be constrained to the interval $[0, 600]$.

5. We have reduced our optimization problem to finding the absolute maximum of the function $A = (1200 - 2l)l = 1200l - 2l^2$ on the interval $[0, 600]$. This function has only one critical point at $l = 300$ as you can verify by solving for the zeros of the derivative dA/dl. At the endpoints $A(0) = 0$, $A(600) = 0$, so $A(300) = 18,000$ yields the maximum area (in square meters) that the farmer can fence off.

To check the reasonableness of our answer, we compare it to the area of a square pasture with three sides made out of a total of $1,200$ m of fencing. We find that the area of the square pasture, $16,000 \text{ m}^2$, is less than the answer we obtained for the maximum area, lending support to our solution of the problem. □

One can solve a typical optimization problem by completing steps that are similar to the ones carried out in the above example:

Procedure for Solving Optimization Problems

1. Read the problem carefully to make sure you understand what the question is. Identify all the quantities involved and name them or designate each one by a symbol. Try drawing a sketch to visualize the problem.

2. Identify the quantity to be optimized and find a mathematical expression for it in terms of the quantities uncovered in Step 1. This is the *objective function* in the terminology of the text.

3. Write down all the relationships you can come up with between the quantities found in step 1.

4. Use the conditions discovered in step 3 to eliminate all but one variable in the function to be optimized. Find the relevant interval for the remaining variable. (For example, by requiring that lengths, areas, and volumes must be positive, etc.)

5. Use calculus (cf. Lesson 17) to find the absolute minimum or maximum value of the function to be optimized, and interpret your result in terms of the original quantities in the problem. Finally, double-check that your answer is reasonable by verifying that it is physically sensible and satisfies the constraints imposed by the problem and, for example, by making a comparison with some easily computable special cases of the problem.

Starter Problems: 3, 4, 5, 7, 11, 19, 32a, 45a.

Recommended Problems: 14b, 17, 21, 24, 26, 45, 47, 53, 59, 61b,c.

Lesson 21 – §4.5: Linear Approximations and Differentials

If a function $f(x)$ is differentiable at $x = a$, then the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

gives the approximation

$$f(x) - f(a) \approx f'(a)(x - a),$$

or

$$f(x) \approx f(a) + f'(a)(x - a),$$

for x near a. The expression

$$L(x) = f(a) + f'(a)(x - a)$$

on the right-hand side is called the *linear approximation* to $f(x)$ at $x = a$.

As you recognize, the graph of the linear approximation is simply the tangent line to the graph of $f(x)$ at $x = a$. Thus, geometrically, using linear approximation amounts to replacing the graph of the function by the graph of the tangent line. As a rule of thumb, the linear approximation gives a good estimate for the values of $f(x)$ when the second derivative $f''(x)$ is small near a.

Example. We use linear approximations to estimate $\sin 0.05$ (Note that the angle measurement is in radians).

Now $f(x) = \sin x$. We need a point near $x = 0.05$ for which we know the precise value for $\sin x$. The obvious choice is $a = 0$. Thus the linear approximation gives

$$L(0.05) = \sin 0 + \cos 0(0.05 - 0) = 0.05.$$

As you can check using your calculator, $\sin 0.05 \approx 0.04998$ to 5 significant figures, so the linear approximation yields a surprisingly good estimate for $\sin 0.05$. But this was more or less expected as $f''(x) = -\sin x$, which is small near the base point $a = 0$. □

Writing $\Delta x = x - a$ for the change in x and $\Delta y = f(x) - f(a)$ for the change in the values of $f(x)$, we can rewrite the expression for linear approximation as

$$\Delta y \approx f'(a)\Delta x.$$

By replacing Δx and Δy by their *differentials* dx, dy, which, in traditional calculus, represent "infinitesimal changes" in x and y, the above formula becomes

$$dy = f'(a)dx.$$

The custom of denoting infinitesimal variations by differentials might seem odd at first, but the notion can be made fully rigorous in the setting of modern differential geometry.

Starter Problems: 2, 4, 7, 9, 11, 13, 23, 35, 41.

Recommended Problems: 16, 18, 25, 30, 31, 39, 49, 57, 58.

Lesson 22 – §4.6: Mean Value Theorem

This section treats an important theorem that can be used to prove rigorously many of the claims justified on intuitive grounds in the previous lessons. Roughly, the theorem asserts that given an interval $[a, b]$ contained in the domain of a differentiable function $f(x)$, there is some c between a and b at which the tangent line is parallel to the line connecting the points $(a, f(a))$ and $(b, f(b))$ on the graph of the function.

The Mean Value Theorem (MVT). Let $f(x)$ be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is some c, $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

Rolle's Theorem is a special case of the MVT obtained by assuming that $f(a) = f(b) = 0$.

Note that for a given $f(x)$ and an interval $[a, b]$, finding the value of c can be difficult if not impossible (consider, for example, $f(x) = \sin x - x^2$, $a = 0$, $b = 1$), but in spite of this, the MVT yields a surprising amount of useful information about the behavior of differentiable functions.

For example, you can now see once and for all without having to appeal to intuitive geometric arguments that if $f'(x) > 0$ on an interval I, then $f(x)$ is strictly increasing on I. Simply choose any a, b in I, $b > a$. Then by the MVT,

$$f(b) - f(a) = f'(c)(b - a), \qquad \text{for some } a < c < b.$$

But as both $f'(c) > 0$ and $b - a > 0$ by assumption, it follows that

$$f(b) - f(a) = f'(c)(b - a) > 0.$$

Consequently $f(b) > f(a)$. But the conclusion holds for any $a < b$ on I, so $f(x)$ must be increasing on I.

The MVT also implies that if two functions have the same derivatives $f'(x) = g'(x)$ on some interval I, then there is a constant C such that $g(x) = f(x) + C$ identically on I. Thus the graph of $g(x)$ can be obtained from the graph of $f(x)$ by a shift in the y-direction.

As you recall (cf. the table on page 31), the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ are the opposite. What do you conclude about the two functions in light of the MVT? Can you derive similar identities between $\tan^{-1} x$ and $\cot^{-1} x$, and $\sec^{-1} x$ and $\csc^{-1} x$?

Starter Problems: 3, 6, 7, 11, 16, 17, 20, 27, 30.

Recommended Problems: 9, 13, 14, 19, 21, 24, 29, 35, 39.

Lesson 23 – §4.7: L'Hôpital's Rule

Suppose that your task is to compute the limit

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x}.$$

The substitution $x = 0$ yields $0/0$, so the limit is in so-called *indeterminate form* and you won't be able to use the quotient rule for limits to solve the problem. As a workaround divide both the numerator and denominator by x and insert $\sin 0 = 0$ in the denominator, after which the original limit problem becomes

$$\lim_{x\to 0} \frac{\dfrac{e^x - 1}{x}}{\dfrac{\sin x - \sin 0}{x}}.$$

Now you can recognize the difference quotients for the functions e^x and $\sin x$ at $x = 0$ in the resulting expression, so taking the limit as $x \to 0$ amounts to computing the derivatives of these functions at $x = 0$. But both derivatives equal 1, so now the quotient rule for limits can be used to compute

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x} = \frac{\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}}{\displaystyle\lim_{x\to 0} \frac{\sin x - \sin 0}{x}} = \frac{\dfrac{d}{dx} e^x \,\big|_{x=0}}{\dfrac{d}{dx} \sin x \,\big|_{x=0}} = 1.$$

The same process can be applied to any limit $\lim_{x\to a} f(x)/g(x)$ in the indeterminate form $0/0$ (that is, $f(a) = g(a) = 0$): If the derivatives $f'(a)$, $g'(a)$ exist and $g'(a) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

This is the basic form of l'Hopital's rule, which is one of the most useful techniques for finding limits of the quotient of two functions in indeterminate form.

Caveat: When putting l'Hospital's rule into practice, you need to make sure that the limit is in indeterminate form, and that you first differentiate $f(x)$ and $g(x)$ separately and only then compute the limit of the quotient of the derivatives.

L'Hopital's rule also applies in the same form to limits $\lim_{x\to a} f(x)/g(x)$ in the indeterminate forms $\pm\infty/\infty$ (that is, the limits of $f(x)$ and $g(x)$ are both either ∞ or

$-\infty$.). In each of the cases $0/0$ and $\pm\infty/\infty$, the limit point a can as well be $\pm\infty$. L'Hopital's rule also holds for one-sided limits with the obvious modifications.

If an application of l'Hopital's rule results in indeterminate form, the method can be applied repeatedly until the value of the limit is found (provided, of course, that the required derivatives exist).

Example. Compute the limit $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

The limit is in indeterminate form $0/0$, so we resort to l'Hopital's rule. By computing derivatives, we get

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}.$$

The resulting limit is still in indeterminate form $0/0$, so to proceed we apply l'Hopital's rule again. This gives

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2},$$

where, in the last step, we were able to compute the limit by substitution. □

L'Hopital's rule can also be used to compare the growth rates of functions at $\pm\infty$. For example,

$$\lim_{x\to\infty}\frac{(\ln x)^n}{x}=\lim_{x\to\infty}\frac{n(\ln x)^{n-1}/x}{1}=\lim_{x\to\infty}\frac{n(\ln x)^{n-1}}{x}=\cdots=\lim_{x\to\infty}\frac{n!}{x}=0,$$

where, in each step save the last, the limit is in indeterminate form ∞/∞. Thus when x approaches infinity, x becomes much larger than $(\ln x)^n$ for any integer n, that is, x *dominates* $(\ln x)^n$.

There are many variants of the basic version of l'Hopital's rule designed to compute the limit in the case of the indeterminate forms $0\cdot(\pm\infty)$, $\infty-\infty$, $1^{\pm\infty}$, 0^0, ∞^0. The instances involving powers typically require an application of the natural logarithm function in their solution.

Example. Compute the limit $\displaystyle\lim_{x\to 0^+}(\sin x)^{\sin x}$.

We write $h(x)=(\sin x)^{\sin x}$, and start by computing

$$\lim_{x\to 0^+}\ln h(x)=\lim_{x\to 0^+}(\sin x)\ln\sin x.$$

The resulting limit is in indeterminate form $0\cdot(-\infty)$, so, in order to apply l'Hopital's rule, we rewrite it as

$$\lim_{x\to 0^+} \ln h(x) = \lim_{x\to 0^+} \frac{\ln \sin x}{\left(\dfrac{1}{\sin x}\right)},$$

which in the form $-\infty/\infty$. We compute derivatives to see that

$$\lim_{x\to 0^+} \ln h(x) = \lim_{x\to 0^+} \frac{\dfrac{\cos x}{\sin x}}{\dfrac{-\cos x}{\sin^2 x}} = \lim_{x\to 0^+} (-\sin x) = 0.$$

Thus the limit of $\ln h(x)$ is 0, so by continuity of the natural exponential function,

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} e^{\ln h(x)} = e^{\lim_{x\to 0^+} \ln h(x)} = e^0 = 1.$$

In conclusion, we have shown that

$$\lim_{x\to 0^+} (\sin x)^{\sin x} = 1.$$

□

Starter Problems: 7, 11, 13, 19, 23, 32, 40, 55, 69.

Recommended Problems: 16, 21, 25, 29, 34, 43, 49, 51, 59, 75, 82, 85, 91.

Mathematics Department Laboratory Manual for Mth 251 –Differential Calculus

Instructions.

1. Overview of laboratory activities. You work in small groups on each laboratory activity according to your recitation instructor's directions. You will spend about 50 minutes of the recitation time on the scheduled lab and the remaining 30 minutes may be used as a question and answer period about homework assignments, for tests, or for other class activities. You are expected to prepare a report observing the format specified for each lab. These reports are due the week following the laboratory activity. Although you collaborate on the laboratory activities in groups during the recitation, your reports should be written up individually. Your instructor and recitation instructor will provide additional information about the expectations and about the grading policy for laboratory activities.

2. Procedure.

Work on and discuss the problems included in the laboratory activity with your group. You should print a paper copy of the lab in advance for recording your work done during the class hour. Complete the assignment and polish up your report outside the class to turn it in to your recitation instructor by the designated deadline.

3. Graphs.

Several of the laboratory activities require you to plot the graph of a function or functions on a given grid. Before graphing, carefully *choose* and *label* the variables and their ranges on your plot. Unlabeled graphs may not be graded. You may need to experiment on your graphing utility to find ranges that correctly display the requisite information.

Laboratory I – Graphing

Background and Goals: This laboratory activity is designed to acquaint you with the graphing features of your calculator or computer and to reinforce the lectures on limits. Specific goals for this activity are:

- **a.** Use calculators or computers to graph functions in different windows.
- **b.** Use calculators or computers to identify limits from a graph and from numerical data.
- **c.** Compute limits analytically to determine the correct answer.

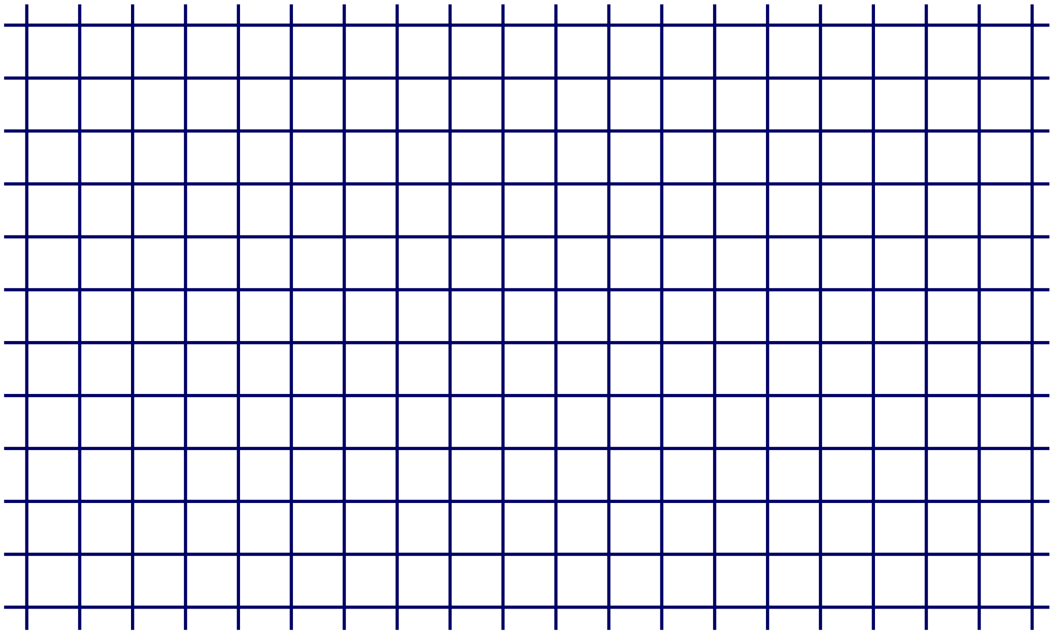
Problem I. This problem is designed to investigate the limit $\lim_{x \to 1} f(x)$ of the function

$$f(x) = \frac{x - 1}{x^2 - 3x + 2}$$

as x approaches 1

a) Graph the function $f(x)$ for x in the interval $[0.5, 1.5]$. Choose an appropriate y-range for your graph. Remember to label all your graphs.

b) Graph the function $f(x)$ for x in the interval $[0.9, 1.1]$. Choose an appropriate y-range for your graph.



c) Based on these two graphs, estimate the limit $\lim_{x \to 1} f(x)$.

d) Fill in the table of values for $f(x)$. Give your answers correct to six decimal places.

x	$f(x)$
0.9	
1.1	
0.99	
1.01	
0.999	
1.001	
0.9999	
1.0001	

e) Estimate the limit $\lim_{x \to 1} f(x)$ based on the values in the table.

f) Compute the limit by factoring the denominator and simplifying the expression for $f(x)$ when $x \neq 1$. What is the limit?

Problem II. This problem is designed to estimate the slope of the tangent line to the graph of $f(x) = \sqrt{x}$ when $x = 1$. Remember to label your graphs.

a) Graph $f(x)$ for x in $[0.5, 1.5]$. Choose an appropriate y-range for your graph.

b) Graph $f(x)$ for x in $[0.9, 1.1]$. Choose an appropriate y-range for your graph.

c) Graph $f(x)$ for x in $[0.99, 1.01]$. Choose an appropriate y-range for your graph.

d) On each of the above graphs, try to draw a tangent line to the graph at the point $(1, 1)$. Estimate the slope of the tangent lines from your sketches.

e) Fill in the chart with some values of the slope of the secant line through the points $(1, 1)$ and $(1 + h, f(1 + h))$ for small values of h. This slope is given by $d(h) = \frac{f(1+h)-f(1)}{h}$. Give your answers correct to six decimal places.

h	$d(h)$
0.1	
0.01	
10^{-10}	
10^{-20}	
10^{-30}	

f) Based on the chart, what does the slope of the tangent line appear to be?

g) Why did you obtain a wrong answer with a calculator? The correct slope is 1/2.

Problem III. Describe in a few sentences what you have learned from this laboratory activity.

Laboratory II – Limits

Background and Goals: This laboratory activity is designed give you more experience in working with limits. The goals are to be able to use graphing calculators or computers to estimate limits from a graph and from numerical data.

Problem I.

a) Why is the equation

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

incorrect?

b) Explain, however, why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is *correct*.

Problem II. Suppose that f satisfies $-10 \leq f(x) \leq 10$ for all x in its domain $I = (0, 1)$.

a) Explain why

$$\lim_{x \to 0^+} \sqrt{x} f(x) = 0$$

by applying the properties of limits from the text. Justify the value of the above limit graphically using the grid below.

b) Let $f(x) = 5 + x$. Use your graphing calculator or computer to find a small interval $(0, \delta)$ with the property that for all values of x in this interval, $\sqrt{x}f(x)$ is within .001 of 0. Draw the graph of the $\sqrt{x}f(x)$ for values of x in this interval on the grid.

Problem III. Consider the function $f(x) = \dfrac{x^2}{(1.001)^x}$ for $x \geq 0$. Try to find the limit $\lim_{x\to\infty} f(x)$ of $f(x)$ as $x \to \infty$, as follows.

a) Chart several values of $f(x)$ for large values of x to estimate the limit. What do you think the limit is?

x	f(x)

b) Graph the function $f(x)$ on your calculator or computer and sketch the graph on the grid for x in the interval $[0, 100]$.

c) Graph the function f on your calculator or computer and sketch the graph on the grid for x in the interval $[0, 1000]$.

d) Graph the function f on your calculator or computer and sketch the graph on the grid for x in the interval $[0, 5000]$.

e) Estimate the limit. How can you be sure that you would not arrive at a different answer by increasing the graphing interval?

Laboratory III – The Intermediate Value Theorem

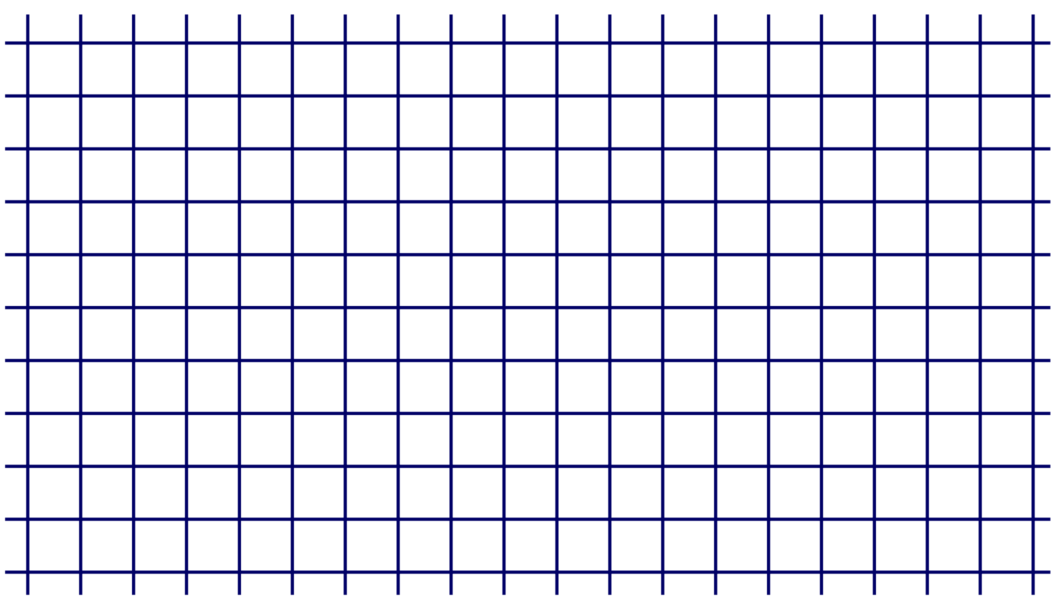
Background and Goals: This laboratory activity is designed to acquaint you with one of the basic theorems in calculus, the Intermediate Value Theorem studied in Lesson 6, and to help you gain appreciation for its applications in both theoretical and concrete problems.

The Intermediate Value Theorem states that given a function $f(x)$ continuous on a closed interval $I = [a, b]$ and a number L between the values $f(a)$ and $f(b)$, then there is at least one c on I so that $f(c) = L$. In short, a continuous function $f(x)$ on a closed interval $I = [a, b]$ realizes every value between $f(a)$ and $f(b)$ on I, or, geometrically, there will be no gaps in the graph of $f(x)$ on I (no matter how close you zoom in).

Problem I. Let $p(x) = x^6 - 3x^2 - x + 1$ be a 6^{th} degree polynomial and let $I = [1, 3]$.

a) Compute $p(1)$ and $p(3)$. Is there a point c on I so that $p(c) = 0$, that is, is there a root of the the polynomial $p(x)$ on I?

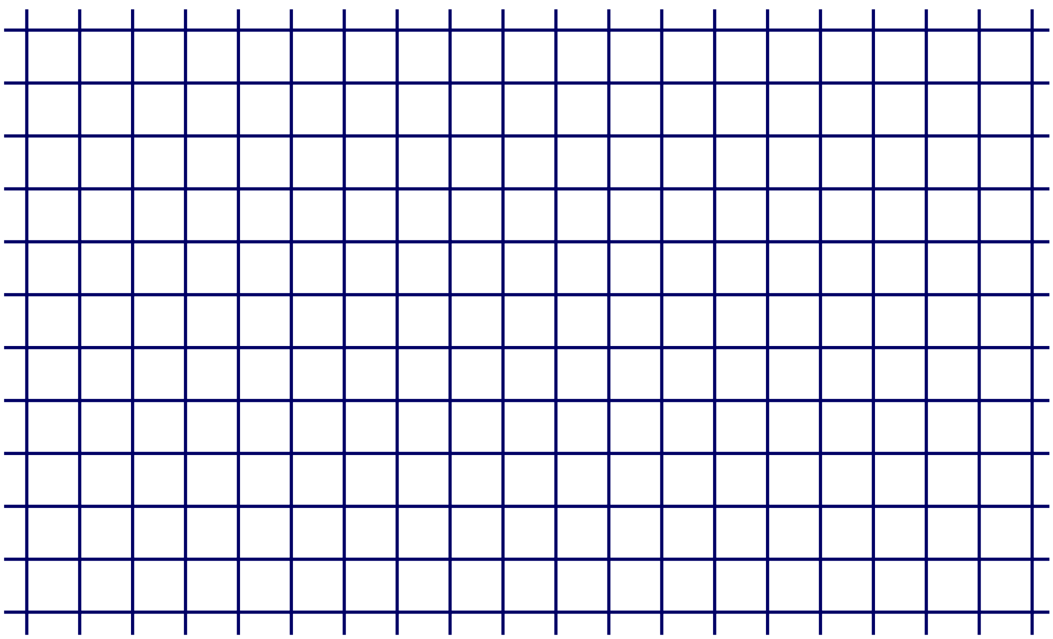
b) Graph the polynomial $p(x)$ on I on the grid below. Does the graph agree with you conclusion in part a?



Problem II. Consider the function $f(x) = \begin{cases} e^{-x}, & \text{if } 0 \leq x \leq 1, \\ x^2 - 5x + 4, & \text{if } 1 < x \leq 3. \end{cases}$

a) Compute $f(0)$ and $f(3)$ and note that these have opposite signs. Next solve the equation $f(x) = 0$ on the interval $I = [0, 3]$. How many solutions did you find? Does your result contradict with the IVT? Explain!

b) Graph $f(x)$ on I on the grid below and verify that the plot agrees with your conclusion in part b.



The IVT is often used to find the rough location of a solution to an equation that can not be solved analytically. This is then used as the *seed*, or the starting value, for a numerical algorithm, such as Newton's method treated in Laboratory Activity IX, for finding an approximation to the solution to high accuracy. But note that the IVT, under certain conditions, only guarantees the existence of a solution on a given interval but does not provide a method for accurately pinpointing its location.

Problem III. Consider the equation $\sin x + x^6 + x = 2$. In order to apply IVT, write $f(x) = \sin x + x^6 + x$. Then finding the roots of the above equation becomes tantamount to finding the x values for which $f(x) = 2$.

a) Compute $f(0)$ and $f(1)$. Can you use the IVT to conclude that the equation has at least one root on the interval $(0, 1)$? Explain!

b) Graph the function $f(x)$ on $[0, 1]$ on the grid below. How many solutions does the equation have?

A simple simple algorithm for locating solutions to equations is furnished by the *bisection method*, which is illustrated in the following problem.

Problem IV. In this problem we try to find an approxiamte solution to the equation $\cos(x^3) - \sin x = 0$ accurate within 0.02.

a) Write $f(x) = \cos(x^3) - \sin x$ so that solving the original equation becomes equivalent to find the zeros of $f(x)$. Why does the IVT apply to $f(x)$ on any (finite) interval?

b) Verify that the equation must have a solution on the interval $[0, 1]$ by computing $f(0)$ and $f(1)$. (Recall that x is expressed in radians.)

c) Next bisect the interval $[0, 1]$ into the subintervals $[0, 0.5]$ and $[0.5, 1]$ and compute $f(0.5)$. Then use IVT to show that our equation must have a solution on $[0.5, 1]$.

d) Next bisect this interval into the subintervals $[0.5, 0.75]$ and $[0.75, 1]$, compute $f(0.75)$. Note that the bisection point 0.75 must a a solution to our equation or, otherwise, the IVT will guarantee the existence of a solution on exactly one of the subintervals. Explain!

e) This process leads to a general algorithm for approximating solutions to equations. Give a step-by-step and detailed description of it.

f) Finally apply the algorithm to find a solution to our original equation accurate within 0.02.

Problem V. The following problems involve typical applications of the IVT.

a) Let $f(x)$ and $g(x)$ be continuous functions on $I = [a, b]$. Suppose that $f(a) < g(a)$ and $f(b) > g(b)$. Show that the graphs of $f(x)$ and $g(x)$ must cross at least once on the interval I.

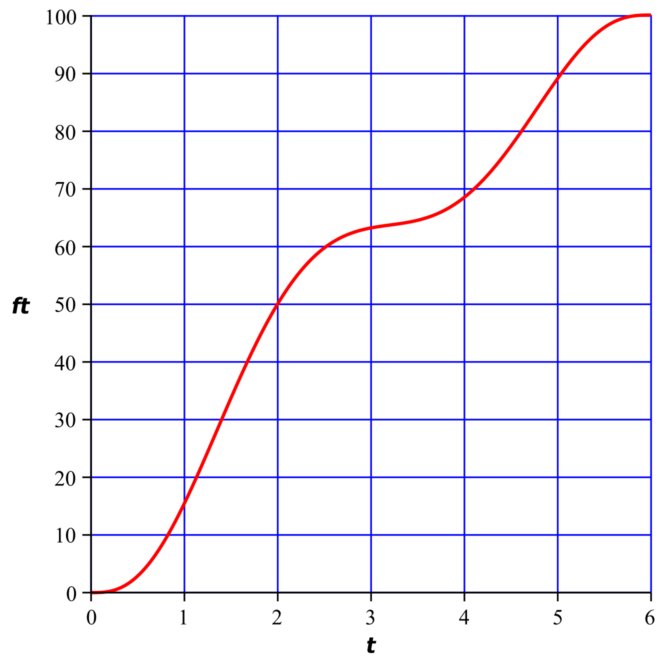
b) Show that the polynomial $p(x) = 32x^5 - 144x^4 - 16x^3 + 456x^2 + 2x - 105$ must have at least five zeros. (Hint: compute $p(0)$, $p(1)$, $p(-1)$, $p(2)$, $p(-2)$,$\ldots$.) Exactly how many distinct zero does $p(x)$ have in light of the fundamental theorem of algebra?

c) A cyclist start at 8 a.m. from Corvallis and rides along highway 20 to Newport, arriving there at 1 p.m. On the next day he returns along the same way, starting from Newport at 8 a.m. and arriving in Corvallis at 1 p.m. Is there a location on highway 20 that the cyclist passed at exactly the same time on both days?

Laboratory IV – Velocity, Tangent Lines

Background and Goals: This laboratory activity is designed to give you more experience in working with the velocity of an object and with the tangent line as an approximation to a function as studied in Lesson 7.

Problem I. A person gets in a car and drives 100 feet to the house next door. The graph below represents the position of the person at various times. Recall that the instantaneous velocity is given by the slope of the tangent line to the curve that plots position against time.



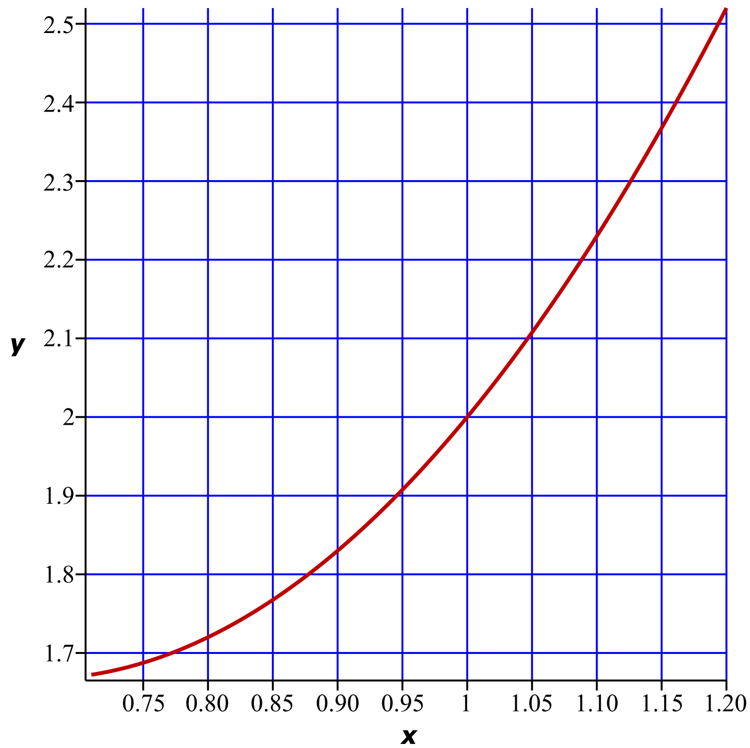
a) How far did the person travel between times 4.5 and 6.0? What is the location of the car at those times?

b) Estimate the velocity of the car at time 2.

c) At which of the *times* 0.5, 1.5, 2.5, 4.0, or 5.5 does the car have the *highest* velocity? Explain.

d) At which of the *times* 0.5, 1.5, 2.5, 4.0 or 5.0 is the acceleration the *highest*? Explain.

Problem II. Consider the function $y = f(x) = 3x^2 - 4x + 3$. The graph of this function near the point $(1, 2)$ is sketched below. Check whether your calculator or computer produces the same graph.



a) On the graph above, sketch the line that seems to you as the best approximation to the function through the point $(1, 2)$.

b) The exact slope of the tangent line is $f'(1) = 6 \cdot 1 - 4 = 2$. Estimate the slope of the line that you sketched in part a . Compare the exact slope of the tangent line with the slope of your line.

c) Any reasonable approximating line to $y = f(x)$ near the point $P = (1, 2)$ must pass through point P. Below, you will try to gather evidence for the fact that the *best* approximating line of this form is the actual tangent line. Write down formulas for the four lines of slope 1.0, 1.5, 2.0, and 2.5 through P.

Line ℓ_1 of slope 1 through P: $y_1 =$

Line ℓ_2 of slope 1.5 through P: $y_2 =$

Line ℓ_3 of slope 2 through P: $y_3 =$

Line ℓ_4 of slope 2.5 through P: $y_4 =$

d) If you use these lines as an approximation to $f(x)$, the error is the absolute value of the difference between $f(x)$ and the corresponding y-value of a point on the line $y = L(x)$:

$$\text{Error}=|f(x) - L(x)|.$$

Derive formulas for the errors using each of the four lines.

Error for line ℓ_1:

$|f(x) - y_1| =$

Error for line ℓ_1:

$|f(x) - y_2| =$

Error for line ℓ_1:

$|f(x) - y_3| =$

Error for line ℓ_1:

$|f(x) - y_4| =$

e) Use the formulas in part d to compute errors for the 4 indicated values of x for each of these lines ℓ_1, ℓ_2, ℓ_3, ℓ_4 by filling in the table below. You should perform all calculations to at least 6 decimal places.

x	error for ℓ_1	error for ℓ_2	error for ℓ_3	error for ℓ_4
1.02				
0.98				
1.001				
0.999				
1.0001				

f) Which of the four lines results in the smallest errors?

Laboratory V – The Chain Rule

Background and Goals: This Laboratory Activity provides hands-on practice on the chain rule studied in Lesson 12.

Problem I. Let

$$f(x) = x^3 - 2x, \qquad g(x) = \frac{1}{x+1}\,.$$

a) Write down an expression for the composition $f(g(x))$.

b) Compute the derivative of $f(g(x))$ with respect to x using the expression in part a.

c) Next find the derivatives $f'(x)$ and $g'(x)$ and then compute the expression $f'(g(x))g'(x)$ according to the chain rule. Did you arrive at the same answer as in part b?

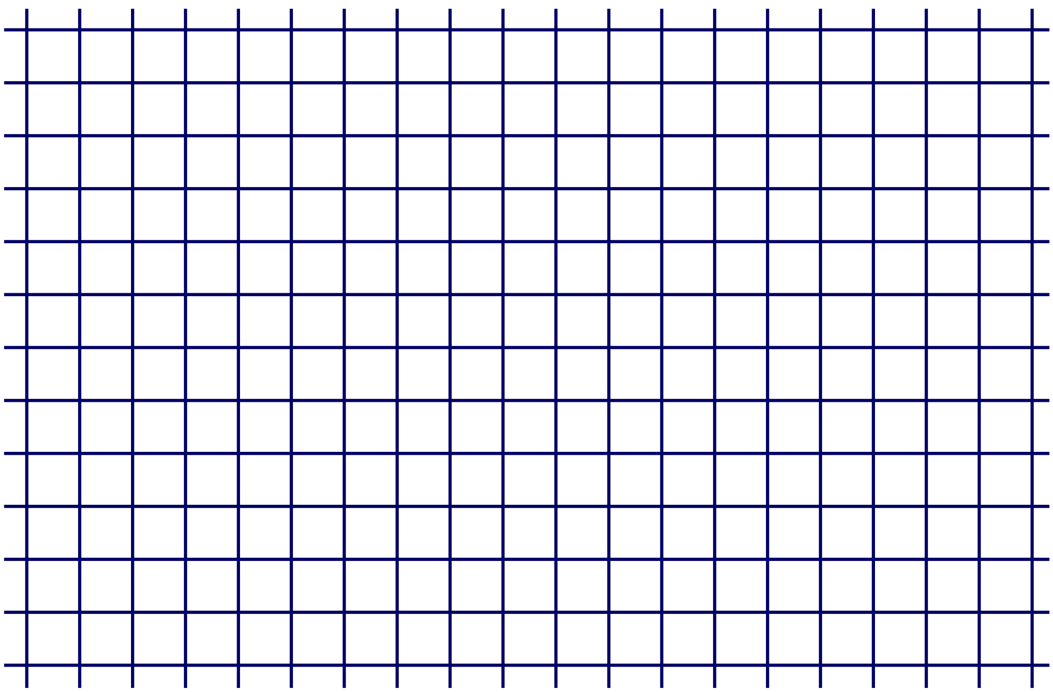
d) Compute the equation of the tangent line to $f(g(x))$ at the point $(1, -7/8)$.

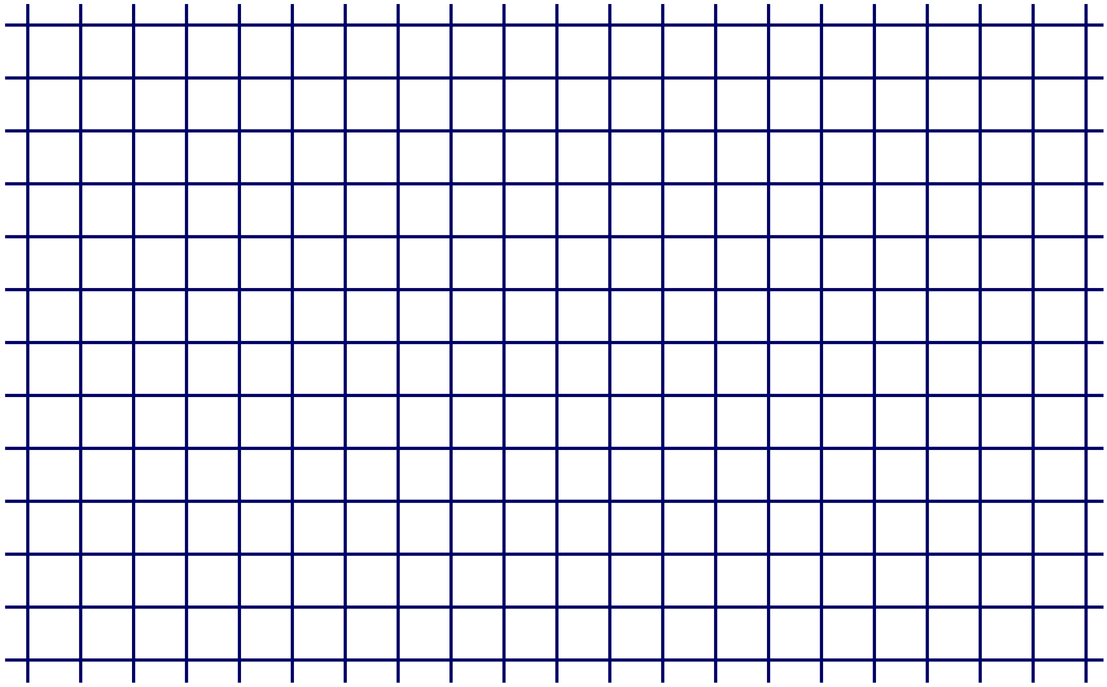
e) Graph both $f(g(x))$ and the tangent line for x in $[0, 2]$ on the grid below. Don't forget to label your graph!

Problem II. Let $f(x) = (x^2 - x)^{\frac{1}{3}}$.

a) Compute the derivative of $f(x)$ by the chain rule.

b) Graph both $f(x)$ and $f'(x)$ on the grids below for x-values between -2 and 2. Choose an appropriate y-range for your graphs.





c) Compare the graphs of f and f' to see if your computation of the derivative seems reasonable. How do you detect intervals on which f is increasing or decreasing?

d) Are there any points at which f is not differentiable?

Laboratory VI - Derivatives in Action

Background and Goals: The main purpose of this Laboratory Activity is to provide more practice on applying the various rules of differentiation.

Problem I. The values of functions $f(x)$ and $g(x)$ and their derivatives at $x = 0, \pm 1, \pm 2$ are collected in the table below.

x	f(x)	f'(x)	g(x)	g'(x)
-2	3	1	-5	8
-1	2	-1	3	2
0	-4	4	-3	9
1	1	5	-1	4
2	3	3	2	7

Compute the following derivatives based on the given data.

a) Let $h(x) = g(x)/\sqrt{9 + \cos 2x}$. Find $h'(0)$.

b) Let $j(x) = 2g(x)(3x^2 + f(x)^2)$. Find $j'(-2)$.

c) Let $k(x) = \pi g(x^2)$. Find $k'(-1)$.

d) Let $l(x) = f(3f(x) - x)$. Find $l'(1)$.

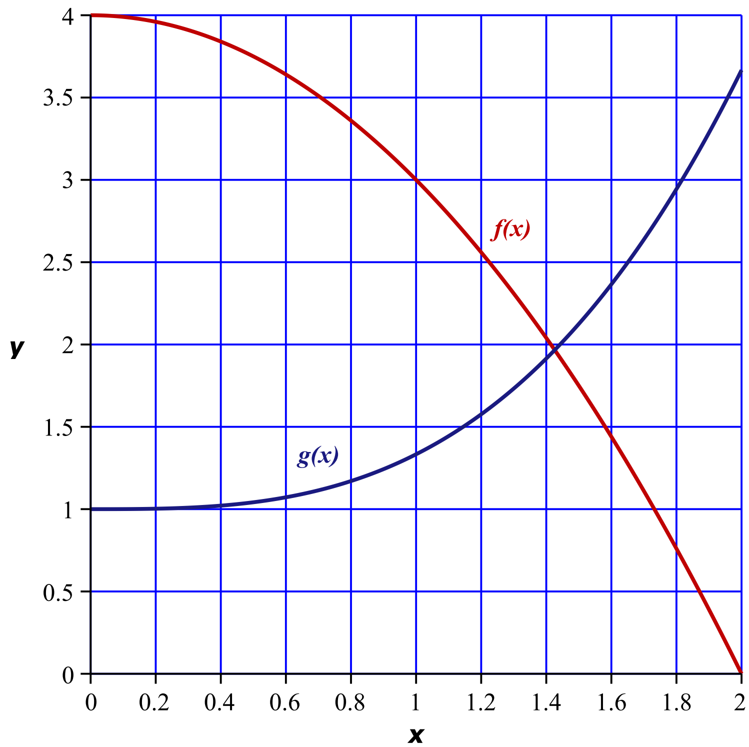
e) Assume that $g(x)$ is also invertible. Let $m(x) = xg^{-1}(x)$. Find $m'(3)$.

f) Let $n(x) = \arctan(4\pi f(x)^2)$. Find $n'(2)$.

g) Suppose $p(x)$ satisfies $p(x)^6 + 4p(x)^3 + f(x)g(x)e^x = 8$. Find $p'(0)$.

h) Let $q(x) = (x^2 - 6x)g(x)$ and suppose that $q'(3) = 10$. Find $g'(3)$.

Problem II. The graphs of functions $f(x)$ and $g(x)$ are depicted below. Using the information from the plot, estimate the values of the given derivatives.



a) $\dfrac{d}{dx}(f(x)g(x))\,|_{x=1}$.

b) $\dfrac{d}{dx}(3g(x) - f(x))\,|_{x=0.6}$.

c) $\frac{d}{dx}(x^4 f(x)) \,|_{x=0.8}$.

d) $\frac{d}{dx}(g(x)\ln(1+x)) \,|_{x=1.4}$.

e) $\frac{d}{dx}\frac{(f(x))^2}{3g(x)-2} \,|_{x=1.8}$.

f) $\frac{d}{dx}\frac{\sqrt{f(x)}}{1+x^2} \,|_{x=0.2}$.

g) $\frac{d}{dx}(g(x))^x \,|_{x=0.4}$.

Problem III. The gas mileage of a car going at speed v (in miles per hour) is given by $M(v)$. Suppose $M(55) = 25$ and $M'(55) = -0.3$.

1) Let $U(v)$ denote the amount of gas the car uses to travel one mile. How are $M(v)$ and $U(v)$ related? Compute $U(55)$ and $U'(55)$.

2) Let $G(v)$ stand for the amount of gasoline the car consumes when it travels at constant speed v for one minute. How are $M(v)$ and $G(v)$ related? Compute $G(55)$ and $G'(55)$.

3) What is the practical significance of the derivatives $M'(55)$, $U'(55)$, and $G'(55)$?

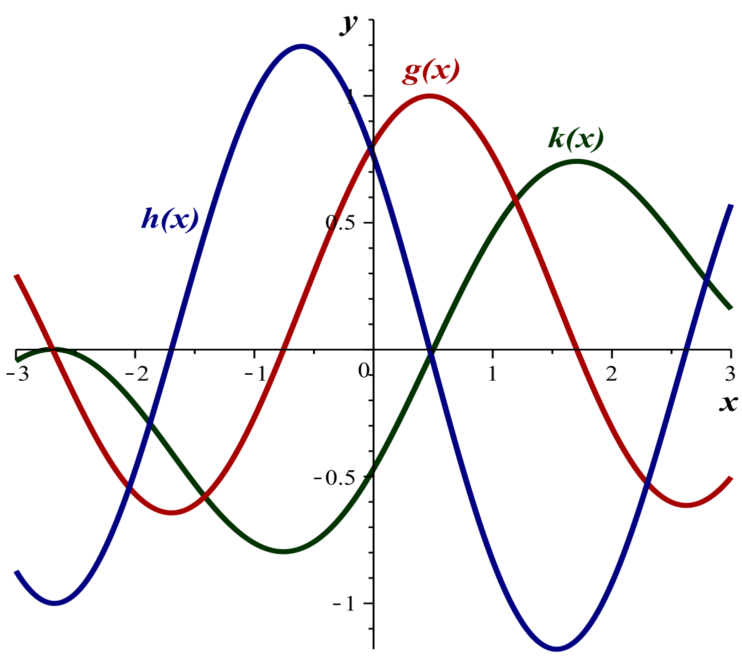
Laboratory VII - Higher Derivatives, Exponential Functions

Background and Goals: This laboratory activity explores the properties of the first and second derivatives of a function, and of exponential functions and their derivatives.

Problem I. Pictured below are the graphs of a function $f(x)$ and the first two derivatives $f'(x)$ and $f''(x)$ for x-values between -3 and 3.

The graph of $g(x)$ is represented by a red curve, and the graphs of $h(x)$ and $k(x)$ by blue and green curves, respectively.

By analyzing where the tangent lines to the various graphs have positive and negative slope, you should be able to determine which of the three function represents $f(x)$, $f'(x)$ or $f''(x)$. Circle the true statement on the next page. For each statement that you deem false, provide a reason, based on the graphs below, for your conclusion.



a) $g(x) = f(x), \qquad h(x) = f'(x) \qquad k(x) = f''(x).$

b) $g(x) = f(x), \qquad k(x) = f'(x) \qquad h(x) = f''(x).$

c) $h(x) = f(x), \qquad g(x) = f'(x) \qquad k(x) = f''(x).$

d) $h(x) = f(x), \qquad k(x) = f'(x) \qquad g(x) = f''(x).$

e) $k(x) = f(x), \qquad h(x) = f'(x) \qquad g(x) = f''(x).$

f) $k(x) = f(x), \qquad g(x) = f'(x) \qquad h(x) = f''(x).$

Problem II.

a) Use a calculator or computer to evaluate the quantity

$$\frac{4^h - 1}{h}$$

for the values 0.1, 0.01, 0.001, 0.0001, and -0.0001 of h. Perform computations correct to five decimal places.

b) Use your answers from part (a) to estimate the limit

$$\lim_{h \to 0} \frac{4^h - 1}{h}$$

to two decimal places.

c) What does the limit in part b represent geometrically?

Problem III.

a) Use a calculator or computer to estimate the limits

$$\lim_{h\to 0} \frac{2.7^h - 1}{h} \qquad \text{and} \qquad \lim_{h\to 0} \frac{2.8^h - 1}{h}$$

to two decimal places. Use a procedure similar to that in problem II. Show your work.

b) What do these limits represent geometrically?

Laboratory VIII - Curve Sketching

Background and Goals: This laboratory activity covers material about concavity, inflection points, and their applications to curve sketching, see Lessons 18 and 19.

Problem I. Consider the function

$$f(x) = \frac{10x(x-1)^4}{(x-2)^3(x+1)^2}.$$

a) Find the x and y-intercepts and all the asymptotes of this function.

b) On the grid below, sketch the graph *by hand* using asymptotes and intercepts, but not derivatives.

c) Use your sketch as a guide to producing a graph (now with the graphing calculator) on the grid below that displays *all* major features of the curve, i.e., asymptotes, intercepts, maxima, minima and inflection points.

d) Use your graph to estimate the maximum and minimum values of $f(x)$.

Problem II. Consider the polynomial $P(x) = x^4 + cx^2 + x$, where c is a constant.

a) For what values of c does the polynomial $P(x)$ have two inflection points? One inflection point? None?

b) Illustrate what you discovered in part (a) by first graphing $P(x)$ with $c = 1, c = 0$, $c = -1$ and $c = -2$, all in the same viewing window on your graphing utility. Then sketch these graphs on the grid below.

c) Describe how the graph changes as c decreases?

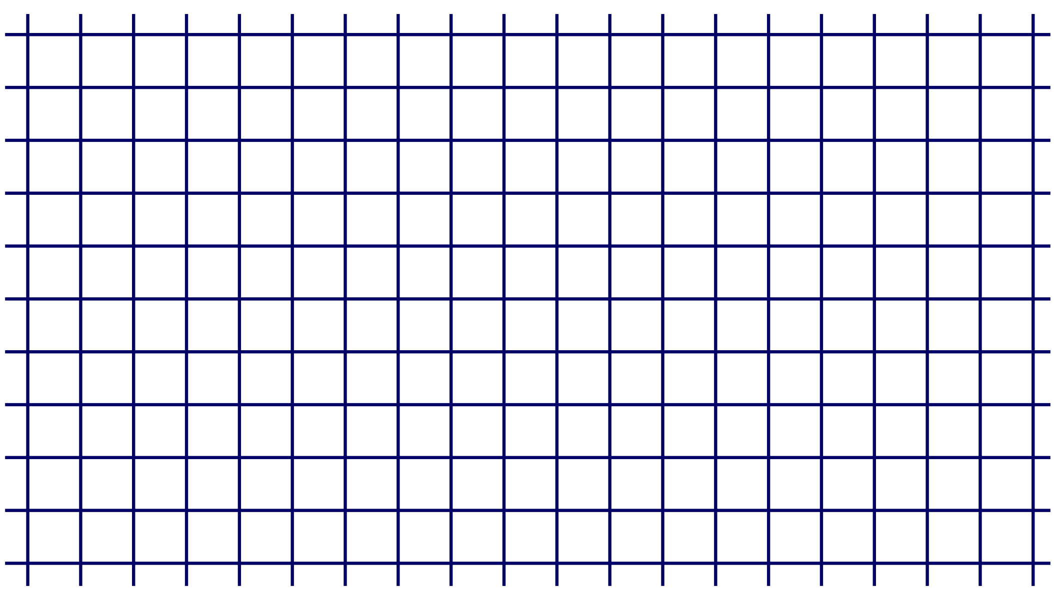
Laboratory IX - Logarithmic Functions, Newton's Method

Background: This laboratory activity covers material about derivatives of logarithmic functions and investigate an algorithm, so-called Newton's method (see section 4.8 of the text), for approximating solutions to equations that can not be solved explicitly.

Problem I. In this problem we investigate linear approximations to the natural logarithm function $\ln x$. Recall that $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$.

a) Find the linear approximation to $f(x) = \ln(x)$ at the point $(1, 0)$. That is, find the equation of the tangent line to the graph of $f(x)$ at that point.

b) Graph both $f(x)$ and the linear approximation on the interval: $.6 \leq x \leq 1.6$. Choose an appropriate y-range for your graph.

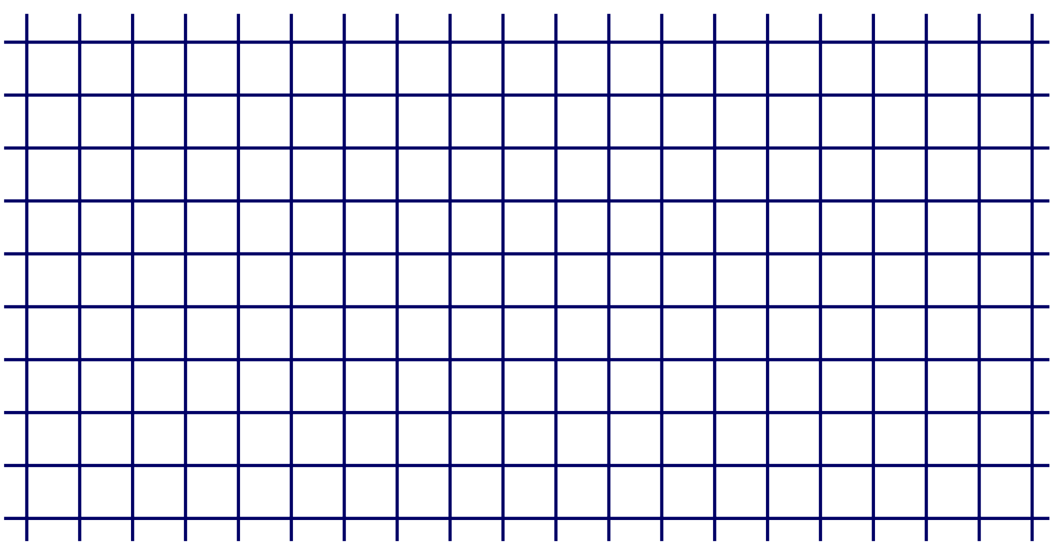


c) For what values of x is the linear approximation accurate to within 0.10? Give an answer accurate to two decimal places. (To determine this, graph the functions $\ln(x)$, $\ln(x) + 0.10$, and your linear approximation in the same viewing window. Then zoom in to the places where the linear approximation meets the graph of $\ln(x) + 0.10$.)

Problem II. Consider the equation $\ln(x-1)-e^{-x} = 0$. Write $f(x) = \ln(x-1)-e^{-x}$ so that the *roots* of the equation correspond to the *zeros* of the function $f(x)$.

a) Can you solve the equation analytically for x? Explain.

b) On the grid below, graph the function $f(x)$ near the point where the graph crosses the x-axis. Choose an appropriate $x-$range and $y-$range for the graph and label the graph. (Your total $x-$range should be 1 to 3 units long.)



c) Use your graph from part (a) to estimate to the nearest tenth where the graph crosses the x-axis. Write x_o for your estimate and compute $f(x_o)$.

d) In order to improve your estimate for the x-intercept of the graph, find the equation for the tangent line to the graph of $f(x)$ at $(x_o, f(x_o))$ and solve for its x-intercept, carrying out the computations on your calculator correct to at least 8 decimals. Write x_1 for the intercept of the tangent line (this is the first iteration in Newton's method). Then compute $f(x_1)$. What do you notice?

e) Next find the tangent line to the graph of $f(x)$ at $(x_1, f(x_1))$ and solve for its x-intercept x_2. Then compute $f(x_2)$. What do you notice?

f) Continue this iterative process to compute x_3, x_4, x_5, and x_6. Then fill out the table below and analyze the results.

n	x_n	$f(x_n)$
0		
1		
2		
3		
4		
5		
6		