Chapter 5: Expansions
Sections 5.1, 5.2, 5.7 & 5.8
1. Power series solutions of ordinary differential equations

- A **power series** about \( x = x_0 \) is an infinite series of the form
  \[
  \sum_{n=0}^{\infty} a_n (x - x_0)^n.
  \]

- This series is **convergent** (or **converges**) if the sequence of partial sums
  \[
  S_n(x) = \sum_{i=0}^{n} a_i (x - x_0)^i
  \]
  has a (finite) limit, \( S(x) \), as \( n \to \infty \). In such a case, we write
  \[
  S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.
  \]

- If the series is not convergent, we say that it is **divergent**, or that it **diverges**.
Radius of convergence

- One can show (Abel’s lemma) that if a power series converges for $|x - x_0| = R_0$, then it converges absolutely for all $x$’s such that $|x - x_0| < R_0$.

- This allows us to define the radius of convergence $R$ of the series as follows:
  - If the series only converges for $x = x_0$, then $R = 0$.
  - If the series converges for all values of $x$, then $R = \infty$.
  - Otherwise, $R$ is the largest number such that the series converges for all $x$’s that satisfy $|x - x_0| < R$.

- A useful test for convergence is the ratio test:

$$R = \frac{1}{K}, \text{ where } K = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|,$$

where $K$ could be infinite or zero, and it is assumed that the $a_n$’s are non-zero.
Power series as solutions to ODE’s

- **Taylor series** are power series.

- A function $f$ is **analytic** at a point $x = x_0$ if it can locally be written as a convergent power series, i.e. if there exists $R > 0$ such that

  $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

  for all $x$’s that satisfy $|x - x_0| < R$.

- If the functions $p/h$ and $q/h$ in the differential equation

  $$h(x)y'' + p(x)y' + q(x) = 0 \quad (1)$$

  are analytic at $x = x_0$, then every solution of (1) is analytic at $x = x_0$. 

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We can therefore look for solutions to (1) in the form of a power series.

**Example:** Solve $y'' - 2y' + y = 0$ by the power series method.

Many special functions are defined as power series solutions to differential equations like (1).

- **Legendre polynomials** are solutions to Legendre’s equation
  $$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$ where $n$ is a non-negative integer.

- **Bessel functions** are solutions to Bessel’s equation
  $$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$ with $\nu \in \mathbb{C}$. 
2. Sturm-Liouville problems

- A **regular Sturm-Liouville problem** is an eigenvalue problem of the form

\[ Ly = -\lambda \sigma(x) y, \quad L y = [p(x)y']' + q(x)y, \]  \hspace{1cm} (2)

where \( p, q \) and \( \sigma \) are real continuous functions on \([a, b], a, b \in \mathbb{R}, p(x) > 0 \) and \( \sigma(x) > 0 \) on \([a, b]\), and \( y(x) \) is square-integrable on \([a, b]\) and satisfies given boundary conditions.

- In what follows, we will use **separated boundary conditions**

\[ C_1 y(a) + C_2 y'(a) = 0, \quad C_3 y(b) + C_4 y'(b) = 0. \]  \hspace{1cm} (3)

- An **eigenvalue** of the Sturm-Liouville problem is a number \( \lambda \) for which there exists an **eigenfunction** \( y(x) \neq 0 \) that satisfies (2) and (3).
One can show that with separated boundary conditions, all eigenvalues of the Sturm-Liouville problem are real (assuming they exist).

In such a case, eigenfunctions associated with different eigenvalues are orthogonal (with respect to the weight function $\sigma$).

Two functions $y_1(x)$ and $y_2(x)$ are orthogonal with respect to the weight function $\sigma$ ($\sigma(x) > 0$ on $[a, b]$) if

$$< y_1, y_2 > \equiv \int_{a}^{b} y_1(x) y_2(x) \sigma(x) \, dx = 0.$$
Legendre’s and Bessel’s equations are examples of **singular Sturm-Liouville problems**.

Legendre’s equation \((1 - x^2)y'' - 2xy' + n(n + 1)y = 0\) can be written as

\[
[p(x)y']' + q(x)y = -\lambda y
\]

where \(p(x) = 1 - x^2\), \(q(x) = 0\) and \(\lambda = n(n + 1)\). In this case there are no boundary conditions and \([a, b] = [-1, 1]\).

Bessel’s equation \(x^2y'' + xy' + (x^2 - \nu^2)y = 0\) can be written in the form (2) by setting \(p(x) = \sigma(x) = x\), \(\lambda = 1\), and \(q(x) = -\nu^2/x\). In this case, \([a, b] = [0, R], R > 0\) and \(y(x)\) is required to vanish at \(x = R\).
3. Orthogonal eigenfunction expansions

- Recall that if $A$ is a square $n \times n$ matrix with real entries, then the (genuine and generalized) eigenvectors of $A$, $U_1, U_2, \cdots, U_n$, form a **basis** of $\mathbb{R}^n$.

- This means that every vector $X \in \mathbb{R}^n$ can be written in the form

$$X = a_1 U_1 + a_2 U_2 + \cdots + a_n U_n,$$

where the coefficients $a_i$ are uniquely determined.

- Moreover, if the $U_i$’s are orthonormal (i.e. orthogonal and of norm one), then each coefficient $a_i$ can be found by taking the dot product of $X$ with $U_i$, i.e. $a_i = < X, U_i >$.

- In this case, (4) is an **orthogonal expansion** of $X$ on the eigenvectors of $A$. 

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Similarly, there exist special linear differential operators, such as Sturm-Liouville operators, whose eigenfunctions form a complete orthonormal basis for a space of functions satisfying given boundary conditions.

We can then use such a complete orthonormal basis, \( \{y_1, y_2, \ldots \} \), to write any function in the space as a uniquely determined linear combination of the basis functions. Such an expansion is called an orthonormal expansion or a generalized Fourier series.

In such a case, for every function \( f \) in the space, we can write

\[
f(x) = \sum_{i=1}^{\infty} a_i y_i(x), \quad a_i = \langle f, y_i \rangle, \quad ||y_i|| = 1.
\]
Trigonometric series

- Trigonometric series are the most important example of Fourier series.

- Consider the Sturm-Liouville problem with periodic boundary conditions \((p(x) = 1, \ q(x) = 0, \ \sigma(x) = 1)\),

\[ y'' + \lambda y = 0, \quad y(\pi) = y(-\pi), \quad y'(\pi) = y'(-\pi). \]

- The eigenfunctions are \(1, \ \cos(x), \ \sin(x), \ \cos(2x), \ \sin(2x), \ \cdots, \ \cos(mx), \ \sin(mx), \ \cdots\), and correspond to the eigenvalues \(0, \ 1, \ 1, \ 4, \ 4, \ \cdots, \ m^2, \ m^2, \ \cdots\).

- The above eigenfunctions are orthogonal but not of norm one. They can be made orthonormal by dividing each eigenfunction by its norm. They form a complete basis of the space of square integrable functions on \([-\pi, \pi]\).