

## The Poincaré Disk Model and The Klein–Beltrami Model

Hyperbolic geometry is a form of non-Euclidean geometry that replaces the parallel postulate of Euclidean geometry. In Euclidean geometry, for two dimensions, the parallel postulate states that for any given line  $l$  and point  $P$  not on  $l$ , there is exactly one line through  $P$  that does not intersect  $l$ , meaning it is parallel to  $l$  (See figure 1). The parallel postulate is false in hyperbolic geometry because there are at least two lines through  $P$ , which do not intersect  $l$ . The parallel postulate has also been proven independent of the other Euclidean axioms because models have been constructed with Euclidean geometry that obeys the axioms of Hyperbolic geometry. There are four models in hyperbolic geometry; Poincaré Disk Model, Klein Model, Poincare half-plane model, and the Hyperboloid [7].

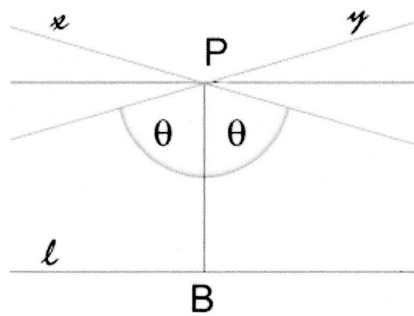


Figure 1 [6]

A model for a form of geometry is an interpretation of the undefined terms and is consistent with the set of axioms for the selected geometry. For my project I'd like to compare and contrast the Poincaré disk model and the Klein model of hyperbolic geometry. They are both models in  $n$ -dimensions, however given two points the segment between them is either a chord, such as in the Klein–Beltrami Model or a circular arc between them as in the Poincaré Model. The two models also differ in the way that you find the distance between two points, or find the angle between two lines. There also exists an isomorphism between the two models.

The first model of hyperbolic geometry was created by Eugenio Beltrami and Felix Klein around 1870 and is known as the Klein–Beltrami Model [1]. In the Klein–Beltrami Model, there exists a disk in the Euclidean plane. In this model, the points are inside of the disk, and the lines are conveyed as chords, which are straight lines with their endpoints on the boundary of the disc (See Figure 2). At the expense of distorting angles, the Klein model preserves straightness [1]. To be considered parallel lines in the Klein model of hyperbolic

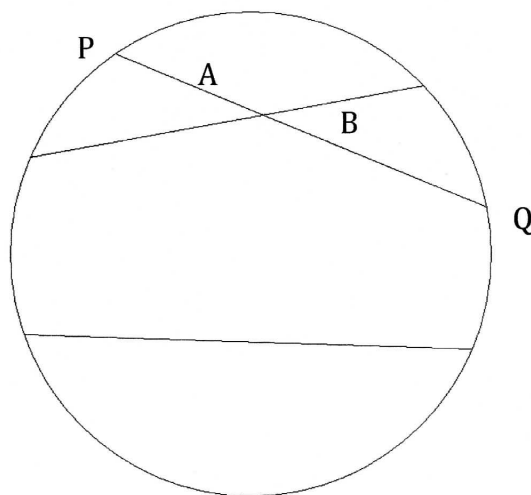


Figure 2 [2]

geometry, the two chords belonging to those lines have to fail to intersect inside the disk.

Whether or not lines are considered perpendicular in the Beltrami-Klein model depends on if one of those lines is a diameter. If one of the lines is a diameter of the disk, then the two lines are hyperbolically perpendicular if and only if in the Euclidean sense they are considered perpendicular. If neither of the lines are diameters, and  $l$  is one line and  $m$  is the other, then they are perpendicular if and only if  $l$  is extended as a Euclidean line and it passes through the pole of  $m$ . The pole of a line is the point where at the endpoints of the line, the tangents to the disk intersect. If the two lines are not perpendicular, then mapping the lines to the Poincaré disk model and thus finding the angle measure, which will be explained later on, will result in the congruent angle measure of the two intersecting lines [3].

To measure the distance between two points,  $A$  and  $B$  in the Klein Model, you must use the ideal endpoints of the line that goes through  $A$  and  $B$ . Ideal endpoints consist of points that lie on the boundary of the disk, therefore are not inside the disk. With ideal endpoints  $P$  and  $Q$ , the distance between  $A$  and  $B$  is demonstrated through the following equation:

$$d(A,B) = \frac{1}{2} |(\log(AB, PQ))|$$
$$= \frac{1}{2} |(\log(AP*BQ)/(BP*AQ))|$$

With, for example,  $AB$  representing the Euclidean distance between point  $A$  and point  $B$ , and  $*$  refers to multiplication between both distances [4].

Henri Poincaré developed the Poincaré Disk model in about 1880 [1]. The Poincaré disk model is an n-dimensional model in hyperbolic geometry where the points are in an n-dimensional disk and there are two types of lines. The first type of line is represented by arcs of a circle in which the ends are perpendicular to the boundary, contained in the circle. The second type of line is a diameter of the circle. Two arcs that never meet are considered to be parallel rays. Perpendicular lines are when two arcs meet orthogonally. In the Poincaré Disk Model, a pair of limit rays is two arcs that meet on the boundary of the circle [2]. The Poincaré Disk Model is built within Euclidean geometry even though it is a model in hyperbolic geometry, which is non-Euclidean [5].

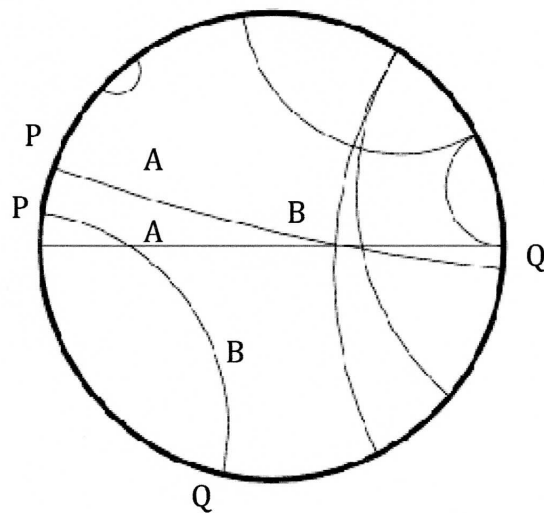


Figure 3 [7]

The distance between two points A and B can be measured using the following equation:

$$d(A,B)=|(\log[AB, PQ])|$$

$$=|(\log(AP*BQ)/(BP*AQ))|$$

where P and Q are the points where the line containing points A and B intersects the boundary. Angles between the rays in a Poincaré Disk Model can be measured directly [2]. To measure the angle between two lines in a Poincaré Disk, you measure the Euclidean angle between their tangent lines [5].

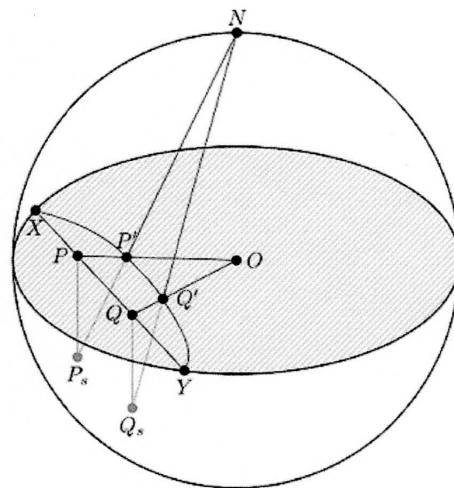


Рис. 14. Связь моделей Клейна и Пуанкаре:  $\xi(P) = P'$ ,  $\xi(Q) = Q'$

Figure 4 [8]

The relationship between the Poincaré Disk model and the Klein model can best be demonstrated best by taking a Klein disk in Euclidean three-space and sitting a sphere on top of the disk (See figure 4). We can then create arcs of circles that are perpendicular to the equator in the sphere by projecting chords of the Klein disk orthogonally upward into the lower hemisphere. If we project that

lower hemisphere back onto the plane in which the Klein disk is sitting, then the equator will create a disk a little larger than the Klein disk. The chords that were projected onto the sphere originally are now Poincaré lines. Through this relationship, lines or angles in the Klein model are congruent if and only if their corresponding lines or angles in the Poincaré model [3].

## Works Cited

- [1] "Non-Euclidean geometry." Encyclopædia Britannica. 2009. Encyclopædia Britannica Online. 25 May. 2009  
<<http://www.britannica.com/EBchecked/topic/417456/non-Euclidean-geometry>>.
- [2] Weisstein, Eric W. "Poincaré Hyperbolic Disk." From *MathWorld*--A Wolfram Web Resource.  
<http://mathworld.wolfram.com/PoincareHyperbolicDisk.html>
- [3] Weisstein, Eric W. "Klein-Beltrami Model." From *MathWorld*--A Wolfram Web Resource. <http://mathworld.wolfram.com/Klein-BeltramiModel.html>
- [4] "The Beltrami-Klein Model."  
<<http://www.math.uncc.edu/~droyster/math3181/notes/hyprgeom/node57.html>>
- [5] "The Poincare Disk." <http://www.calvin.edu/~venema/eeg/eeg-14.pdf>
- [6] "Hyperbolic Geometry." [http://en.wikipedia.org/wiki/Hyperbolic\\_geometry](http://en.wikipedia.org/wiki/Hyperbolic_geometry)
- [7] [http://en.wikipedia.org/wiki/File:Klein\\_model.png](http://en.wikipedia.org/wiki/File:Klein_model.png)
- [8] [http://farm4.static.flickr.com/3157/2548960134\\_054a3a8d25.jpg?v=0](http://farm4.static.flickr.com/3157/2548960134_054a3a8d25.jpg?v=0)