First of all, it is easy to show from (46) that:

\[
\begin{align*}
\nabla \times A'(r, t) &= \nabla \times A(r, t) \\
- \nabla U'(r, t) - \frac{\partial}{\partial t} A'(r, t) &= - \nabla U(r, t) - \frac{\partial}{\partial t} A(r, t)
\end{align*}
\]

(47)

Any gauge, \( \{ A', U' \} \), which satisfies (46) therefore yields the same electric and magnetic fields as \( \{ A, U \} \).

Conversely we shall show that if two gauges, \( \{ A, U \} \) and \( \{ A', U' \} \), are equivalent, there must exist a function \( \chi(r, t) \) which establishes relations (46) between them. Since, by hypothesis:

\[
B(r, t) = \nabla \times A(r, t) = \nabla \times A'(r, t)
\]

(48)

we have:

\[
\nabla \times (A' - A) = 0
\]

(49)

This implies that \( A' - A \) is the gradient of a scalar function:

\[
A' - A = \nabla \chi(r, t)
\]

(50)

\( \chi(r, t) \) is, for the moment, determined only to within an arbitrary function of \( t, f(t) \). Furthermore, the fact that the two gauges are equivalent means that:

\[
E(r, t) = - \nabla U(r, t) - \frac{\partial}{\partial t} A(r, t) = - \nabla U'(r, t) - \frac{\partial}{\partial t} A'(r, t)
\]

(51)

that is:

\[
\nabla (U' - U) + \frac{\partial}{\partial t} (A' - A) = 0
\]

(52)

According to (50), we must have:

\[
\nabla (U' - U) = - \nabla \frac{\partial}{\partial t} \chi(r, t)
\]

(53)

Consequently, the functions \( U' - U \) and \( - \frac{\partial}{\partial t} \chi(r, t) \) can differ only by a function of \( t \); thus, we can choose \( f(t) \) so as to make them equal:

\[
U' - U = - \frac{\partial}{\partial t} \chi(r, t)
\]

(54)

This completes the determination of the function \( \chi(r, t) \) (to within an additive constant). Two equivalent gauges must therefore satisfy relations of the form (46).

\[ \beta. \] Equations of motion and the Lagrangian

In the electromagnetic field, the charged particle is subject to the Lorentz force:

\[
F = q [E + \mathbf{v} \times \mathbf{B}]
\]

(55)
where $v$ is the velocity of the particle at the time $t$. Newton's law therefore gives the equations of motion in the form:

$$m \ddot{\mathbf{r}} = q \left[ \mathbf{E}(r, t) + \mathbf{v} \times \mathbf{B}(r, t) \right]$$

(56)

Projecting this equation onto $\mathbf{Ox}$ and using (45), we obtain:

\[
\begin{aligned}
m \ddot{x} &= q \left[ E_x + v_y B_z - v_z B_y \right] \\
&= q \left[ -\frac{\partial U}{\partial x} - \frac{\partial A_y}{\partial t} + y \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - z \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \right]
\end{aligned}
\]

(57)

It can easily be shown that these equations can be derived from the Lagrangian by using (15):

\[
\mathcal{L}(r, \mathbf{v}, t) = \frac{1}{2} m \dot{r}^2 + q \mathbf{v} \cdot \mathbf{A}(r, t) - qU(r, t)
\]

(58)

Therefore, although the Lorentz force is not derived from a potential energy, we can find a Lagrangian for the problem.

Let us show that Lagrange's equations (15) do yield the equations of motion (56), using the Lagrangian (58). To do so, we shall first calculate:

\[
\frac{\partial \mathcal{L}}{\partial \dot{x}} = m \ddot{x} + q A_x(r, t)
\]

(59)

\[
\frac{\partial \mathcal{L}}{\partial \dot{y}} = q \dot{y} \frac{\partial}{\partial x} A_x(r, t) - q \frac{\partial}{\partial x} U(r, t)
\]

(60)

Lagrange's equation for the $x$-coordinate can therefore be written:

\[
\frac{d}{dt} \left[ m \ddot{x} + q A_x(r, t) \right] - q \dot{y} \frac{\partial}{\partial x} A_x(r, t) + q \frac{\partial}{\partial x} U(r, t) = 0
\]

(61)

Writing this equation explicitly and using (16), we again get (57):

\[
m \ddot{x} + q \left[ \frac{\partial A_x}{\partial t} + \dot{y} \frac{\partial A_x}{\partial y} + z \frac{\partial A_x}{\partial z} \right] - q \left[ \dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_x}{\partial y} + z \frac{\partial A_x}{\partial z} \right] + q \frac{\partial U}{\partial x} = 0
\]

(62)

that is:

\[
m \ddot{x} = q \left[ -\frac{\partial U}{\partial x} - \frac{\partial A_x}{\partial t} + y \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - z \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \right]
\]

\[
Y. \quad \text{Momentum. The classical Hamiltonian}
\]

The Lagrangian (58) enables us to calculate the conjugate momenta of the cartesian coordinates $x, y, z$ of the particle. For example:

\[
p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x} + q A_x(r, t)
\]

(63)
APPENDIX III

The momentum of the particle, which is, by definition, the vector whose components are \( (p_x, p_y, p_z) \), is no longer equal, as in (19), to the mechanical momentum \( m \mathbf{v} \):

\[
\mathbf{p} = m \mathbf{\dot{r}} + q \mathbf{A}(r, t)
\]  

(64)

Finally, we shall write the classical Hamiltonian:

\[
\mathcal{H}(r, \mathbf{p}; t) = \mathbf{p} \cdot \mathbf{\dot{r}} - L
\]

\[
- \mathbf{p} \cdot \frac{1}{m} (\mathbf{p} - q \mathbf{A}) - \frac{1}{2m} (\mathbf{p} - q \mathbf{A})^2 - \frac{q}{m} (\mathbf{p} - q \mathbf{A}) \cdot \mathbf{A} + qU
\]  

(65)

that is:

\[
\mathcal{H}(r, \mathbf{p}; t) = \frac{1}{2m} [\mathbf{p} - q \mathbf{A}(r, t)]^2 + qU(r, t)
\]  

(66)

COMMENT:

Hamiltonian formalism therefore uses the potentials \( \mathbf{A} \) and \( U \), and not the fields \( \mathbf{E} \) and \( \mathbf{B} \) directly. The result is that the description of the particle depends on the gauge chosen. It is reasonable to expect, however, since the Lorentz force is expressed in terms of the fields, that predictions concerning the physical behavior of the particle must be the same for two equivalent gauges. The physical consequences of the Hamiltonian formalism are said to be gauge-invariant. The concept of gauge invariance is analyzed in detail in complement H III.

5. The principle of least action

Classical mechanics can be based on a variational principle, the principle of least action. In addition to its theoretical importance, the concept of action serves as the foundation of the Lagrangian formulation of quantum mechanics (cf. complement I III). This is why we shall now briefly discuss the principle of least action and show how it leads to Lagrange's equations.

a. GEOMETRICAL REPRESENTATION OF THE MOTION OF A SYSTEM

First of all, consider a particle constrained to move along the \( \hat{O}x \) axis. Its motion can be represented by tracing, in the \( (x, t) \) plane, the curve defined by the law of motion which yields \( x(t) \).

More generally, let us study a physical system described by \( N \) generalized coordinates \( q_i \); (for an \( n \)-particle system in three-dimensional space, \( N = 3n \)). It is convenient to interpret the \( q_i \) to be the coordinates of a point \( Q \) in an \( N \)-dimensional Euclidean space \( R_N \). There is then a one-to-one correspondence between the positions of the system and the points of \( R_N \). With each motion of the system is associated a motion of point \( Q \) in \( R_N \), characterized by the \( N \)-dimensional vector function \( Q(t) \) whose components are the \( q_i(t) \). As in the simple case of a single particle moving in one dimension, the motion of point \( Q \), that is, the motion of the system, can be represented by the graph of \( Q(t) \), which is a curve in an \((N + 1)\)-dimensional space-time (the time axis is added to the \( N \) dimensions of \( R_N \)). This curve characterizes the motion being studied.