QM II
Phys 652

Lecture #8



## Rotations in the state space

$$|\Psi\rangle \Rightarrow |\Psi'\rangle = \mathcal{D}(R)|\Psi\rangle$$

Drehung ← rotation (German)

Need to preserve $\langle\Psi|\Psi\rangle = \langle\Psi'|\Psi'\rangle$

$\Downarrow$

$\mathcal{D}(R)$ must be unitary! ← Recall Lectures #10-11 of Phys 651

$\Downarrow$

$$\mathcal{D}(R)\underbrace{\mathcal{D}^{+}(R)}_{\mathcal{D}^{-1}(R)} = I$$

From Phys 651 $\Rightarrow$ unitary transformations

$$U_{\varepsilon} = 1 - iG\varepsilon$$

$G$ ← generator

angular momentum → $G = \frac{J_z}{\hbar}$, $\varepsilon = d\varphi$ $\Downarrow$ rotation around z-axis by $d\varphi$

$G = \frac{H}{\hbar}$, $\varepsilon = dt$ $\Downarrow$ time evolution by $dt$

$G = \frac{P_x}{\hbar}$, $\varepsilon = dx$ $\Downarrow$ translation by $dx$

$\Rightarrow$

$$\mathcal{D}(\vec{n}, d\varphi) = 1 - i\left(\frac{\vec{J}\cdot\vec{n}}{\hbar}\right)d\varphi$$

rotation around $\vec{n}$ by $d\varphi$

or, in the case of finite rotations ⇒

$$\mathcal{D}(\vec{n}, \varphi) = \underbrace{\left(1 - \frac{i}{\hbar}\vec{J}\cdot\vec{n}\,\underbrace{d\varphi}_{=\frac{\varphi}{N}}\right)\left(1 - i\frac{\vec{J}\cdot\vec{n}}{\hbar}d\varphi\right)\cdots}_{N \text{ times}} =$$

$$= \lim_{N\to\infty}\left[1 - i\frac{\vec{J}\cdot\vec{n}}{\hbar}\frac{\varphi}{N}\right]^N = \exp\left(-\frac{i}{\hbar}(\vec{J}\cdot\vec{n})\varphi\right)$$

What happens to observables upon rotation? ⇒
Lecture #10 of Phys 657 ⇒

$$\underbrace{A' = \mathcal{D}(R)\, A\, \mathcal{D}^+(R), \quad A = \mathcal{D}^+(R) A' \mathcal{D}(R)}_{\text{transformation law}}$$

Let's apply infinitesimal rotation and see how A changes:

$$A' = \left(1 - \frac{i}{\hbar}(\vec{J}\cdot\vec{n})d\varphi\right) A \left(1 + \frac{i}{\hbar}(\vec{J}\cdot\vec{n})d\varphi\right) =$$

$$= A - \frac{i}{\hbar}d\varphi\left[\vec{J}\cdot\vec{n}, A\right] = A'$$

neglect $O((d\varphi)^2)$

<u>Definition</u>; an observable is <u>scalar</u> if $A = A'$,
i.e. $[\vec{J}\cdot\vec{n}, A] = 0$

If an observable commutes with $\vec{J}\cdot\vec{n}$
(i.e. generator of rotation about $\vec{n}$-axis),
it does <u>not</u> change upon rotation about $\vec{n}$.

What if $A = H$ ← Hamiltonian of an isolated system

Consider an isolated system in a state $|\Psi(t_0)\rangle$. Under rotation, the state transforms into $|\Psi'(t_0)\rangle = \mathcal{D}(R)\,|\Psi(t_0)\rangle$

At a time $t_0 + dt$, the state is

$$|\Psi'(t_0+dt)\rangle = |\Psi'(t_0)\rangle + dt\cdot\left.\frac{d|\Psi'\rangle}{dt}\right|_{t_0}$$

$$= |\Psi'(t_0)\rangle + \frac{dt}{i\hbar}H|\Psi'(t_0)\rangle = \mathcal{D}(R)\,|\Psi(t_0+dt)\rangle$$

$i\hbar\frac{\partial\Psi'}{\partial t} = H\Psi'$ (9.1)

If we had <u>not</u> performed rotation ⇒ the state at $t_0 + dt$ would be:

$$|\Psi(t_0+dt)\rangle = |\Psi(t_0)\rangle + \frac{dt}{i\hbar}H|\Psi(t_0)\rangle$$

Now rotate ⇒ apply $\mathcal{D}(R)$ to ⇒

$$\underbrace{\mathcal{D}(R)|\Psi(t_0+dt)\rangle}_{|\Psi'(t_0+dt)\rangle} = \underbrace{\mathcal{D}(R)|\Psi(t_0)\rangle}_{|\Psi'(t_0)\rangle} +$$

$$+\frac{dt}{i\hbar}\mathcal{D}(R)H|\Psi(t_0)\rangle \qquad (9.2)$$

Compare (9.1) and (9.2) $\Rightarrow$

$$H\underbrace{|\Psi'(t_0)\rangle}_{\mathcal{D}(R)|\Psi(t_0)\rangle} = \mathcal{D}(R)H|\Psi(t_0)\rangle \Rightarrow \quad [H, \mathcal{D}(R)] = 0$$

$$\Updownarrow$$

H is a scalar observable $\Leftarrow$ $[H, \vec{J}\cdot\vec{n}] = 0$

$\uparrow$

does <u>not</u> change upon rotation

From Phys 657 $\Rightarrow$ time evolution of the mean value of an observable $A$ $\Rightarrow$

$$\frac{d\langle A\rangle}{dt} = \left\langle\frac{\partial A}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle[A,H]\rangle$$

If $A \equiv \vec{J} \Rightarrow \quad \frac{d\langle\vec{J}\rangle}{dt} = \frac{1}{i\hbar}\langle[\vec{J},H]\rangle = 0$

total angular momentum of an isolated system is a constant of motion: conservation of angular momentum is a consequence of rotational invarience. (5)

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Recall Lecture # ~~8~~ (on geometric rotations)

$$[R_x(\varepsilon), R_y(\varepsilon)] = R_z(\varepsilon^2) - 1$$

$\Downarrow$ QM $\Leftrightarrow \mathcal{D}(\vec{n}, d\varphi)$

$$\left(1 - \frac{i}{\hbar} J_x \varepsilon - \frac{J_x^2 \varepsilon^2}{2\hbar^2}\right)\left(1 - i\frac{J_y}{\hbar}\varepsilon - \frac{J_y^2\varepsilon^2}{2\hbar^2}\right) -$$

keep $O(\varepsilon^2)$

$$-(\text{same}, y \leftrightarrow x) = \cancel{1} - \frac{i}{\hbar} J_z \varepsilon^2 - \cancel{1} \overset{\text{WS!}}{\Longrightarrow}$$

HW! ← details

$\Downarrow$

generalizing to other axes $\Leftarrow$ $[J_x, J_y] = i\hbar J_z$

$\Downarrow$

$$[J_i, J_y] = i\hbar \varepsilon_{ijk} J_k$$

↑ Levi-Civita tensor

$$\varepsilon_{ijk} = \begin{cases} +1 & \leftarrow \text{even permutation} \\ -1 & \leftarrow \text{odd permutation} \\ 0 & \leftarrow \text{otherwise} \end{cases}$$

## So, how would group theory help?

$\Downarrow$

We could have avoided all derivations above!

## Noether's theorem:

For every continuous symmetry there exists a corresponding conservation law, i.e. there exists a conserved observable

$\Downarrow$

Group $\Longrightarrow$ unitary representation $U = e^{i\varepsilon G}$

Wigner theorem: for every symmetry group there is a unitary representation $U = e^{i\varepsilon G}$ ← generator

$\Downarrow$

$|\Psi'\rangle = U|\Psi\rangle$

$\Downarrow$

$H' = U(\varepsilon) H U^{-1}(\varepsilon) = H$ ← Hamiltonian is invariant under this unitary transformation

$\Downarrow$

$[H, G] = 0 \Rightarrow$ G is a constant of the motion

$\Downarrow$

$[G_i, G_k] = C_{ikl} G_l$ ← $C_{ikl}$ depends on G $\Rightarrow$ (if $G \equiv J \Rightarrow C_{ikl} \equiv \varepsilon_{ikl}$; $G \equiv P \Rightarrow C_{ikl} = 0$)

$\Downarrow$

Generators of a symm. group are conserved observables whose commutation relations are uniquely determined by group structure