QM II
Phys 652

Lecture #2

Ladder operators
Spherical harmonics

Last time: $\vec{L}^2 |l,m\rangle = \hbar^2 l(l+1)|l,m\rangle$

$l \geq 0$ $\quad L_z |l,m\rangle = \hbar \underbrace{m}_{=0,\pm1,\pm2,\ldots} |l,m\rangle$

$Y_l^m(\theta,\varphi) = \langle \vec{n} | l,m\rangle$

Introduce ladder operators $L_\pm = L_x \pm i L_y$

From HW #1 $\Rightarrow$ $L_\mp L_\pm = \vec{L}^2 - L_z^2 \mp \hbar L_z$

$\Downarrow$

$\langle l,m | \vec{L}^2 - L_z^2 \mp \hbar L_z | l,m\rangle = \hbar^2 l(l+1) -$

$- \hbar^2 m^2 \mp \hbar^2 m = \hbar^2 \left(l(l+1) - m(m\pm1)\right) =$

$= \hbar^2 (l \mp m)(l \pm m + 1)$

From another side, $\quad L_\pm^+ = L_\mp$

$\langle l,m | L_\mp L_\pm | l,m\rangle = \langle l,m | L_\pm^+ L_\pm | l,m\rangle =$

$= \| L_\pm |l,m\rangle \|^2 \geq 0 \Rightarrow \underline{(l \mp m)(l \pm m+1) \geq 0}$

show $\Rightarrow$

$\boxed{-\ell \leq m \leq \ell}$ ← prove this to yourself!!

When $m = \ell \Rightarrow \| L_+ |\ell, \ell\rangle \| = 0$

$$\Downarrow$$

$$\underline{L_+ |\ell, \ell\rangle = 0}$$

Similarly, $\underline{L_- |\ell, -\ell\rangle = 0}$

Since $[\vec{L}^2, L_i] = 0 \Rightarrow [\vec{L}^2, L_\pm] = 0 \Rightarrow$

$\Rightarrow \vec{L}^2 \underbrace{L_\pm |\ell, m\rangle}_{= |\psi\rangle} = L_\pm \vec{L}^2 |\ell, m\rangle = \hbar^2 \ell(\ell+1) \cdot$

$\cdot \underbrace{L_\pm |\ell, m\rangle}_{= |\psi\rangle} \Rightarrow \vec{L}^2 |\psi\rangle = \hbar^2 \ell(\ell+1) |\psi\rangle$

$$\Downarrow$$

So, $L_\pm$ operators acting on a state do <u>not</u> change ℓ

What about m? $\Rightarrow$

$L_z L_\pm |\ell, m\rangle = ?$

From HW #1 $\Rightarrow [L_z, L_\pm] = \pm \hbar L_\pm \Rightarrow$

$L_z L_\pm |\ell, m\rangle = L_\pm \underbrace{L_z |\ell, m\rangle}_{= \hbar m |\ell, m\rangle} \pm \hbar L_\pm |\ell, m\rangle =$

$= \hbar(m\pm 1)\underbrace{L_\pm|l,m\rangle}_{\overset{\shortparallel}{|\psi\rangle}} \Rightarrow L_z|\psi\rangle =$

$= \hbar(m\pm 1)|\psi\rangle$

$\Updownarrow$

$|\psi\rangle = \underset{\substack{\uparrow \\ \text{const}}}{C_\pm} |l, m\pm 1\rangle$

$L_\pm$ change the state from $|l,m\rangle$ to $|l, m\pm 1\rangle$

From page 1 $\Rightarrow \| L_\mp |l,m\rangle\|^2 = \hbar^2(l\pm m)\cdot(l\mp m+1) = |C_\mp|^2 \Rightarrow$

$L_\pm |l,m\rangle = \hbar\sqrt{(l\mp m)(l\pm m+1)}\,|l, m\pm 1\rangle$

$\Updownarrow$ project on $|\vec{n}\rangle$:

$L_\pm Y_l^m(\theta,\varphi) = \hbar\sqrt{(l\mp m)(l\pm m+1)}\,Y_l^{m\pm 1}(\theta,\varphi)$

<u>Morning coffee exercise</u> (Sakurai, p. 200 ← 2nd edition):

$L_+ \doteq -i\hbar e^{i\varphi}\left[i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right]$ ← show!

Then, $L_+|l,l\rangle = 0 \Leftrightarrow L_+ Y_l^l(\theta,\varphi) = 0 =$ differential equation $\Rightarrow$ can solve for Y_l^l

Show that $Y_\ell^\ell(\theta,\varphi) = C_\ell e^{i\ell\varphi} \sin^\ell\theta$ (Sakurai 3.6.34)

Apply $L_-|\ell,\ell\rangle \Rightarrow |\ell, m-1\rangle$

$$\Updownarrow$$

Show that $\Leftarrow Y_\ell^{m-1}(\theta,\varphi)$

$$Y_\ell^m(\theta,\varphi) = \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}}\, e^{im\varphi} \underbrace{\frac{1}{\sin^m\theta} \cdot \frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}}(\sin\theta)^{2\ell}}_{P_\ell^m(\cos\theta)}$$

$P_\ell^m(\cos\theta)$ associated Legendre polyns. at $m>0$

and $Y_\ell^{-m}(\theta,\varphi) = (-1)^m [Y_\ell^m(\theta,\varphi)]^*$

Summary: a state $|\ell, m\rangle$ is an eigenstate of $\vec{L}^2$ and L_z, where $\ell \geq 0$ and $|m| \leq \ell$

m integer $\Rightarrow$ ℓ is integer

Still need to figure out why $\lambda = \hbar^2 \ell(\ell+1)$?

λ from Lectures # 1 and 2

$\hbar$ dimensionality

Consider an arbitrary eigenstate of $\vec{J}^2$ and J_z ⑤

$|j, m_j\rangle$

not necessarily orbital angular momentum $\Rightarrow$ generalized angular momentum

Apply $J_z J_+ |j, m_j\rangle = (J_+ J_z + \hbar J_+) |j, m_j\rangle =$

$= J_+ \hbar m_j |j, m_j\rangle + \hbar J_+ |j, m_j\rangle =$

$= \hbar (m_j + 1) J_+ |j, m_j\rangle$

Also, $\vec{J}^2 J_+ |j, m_j\rangle = J_+ \vec{J}^2 |j, m_j\rangle =$

$= \hbar^2 \underbrace{j(j+1)}_{\lambda} \underbrace{J_+ |j, m_j\rangle}_{= C_+ |j, m_j + 1\rangle}$

let's say this is λ

say we don't know that $\lambda = j(j+1)$

Consider $\langle j, m_j | \vec{J}^2 - J_z^2 | j, m_j\rangle =$

$= \langle j, m_j | J_x^2 + J_y^2 | j, m_j\rangle \geq 0 \Rightarrow \lambda - m_j^2 \geq 0$

then, there is $\Leftarrow \lambda \geq m_j^2$

some m_j^{max} for which $J_+ |j, m_j^{max}\rangle = 0$

Since $J_- J_+ = \vec{J}^2 - J_z^2 - \hbar J_z \Rightarrow$

$$J_-\left(\underbrace{J_+|j,m_j^{max}>}_{=0}\right)=0 \Rightarrow$$

$$\left(\vec{J}^2-J_z^2-\hbar J_z\right)|j,m_j^{max}>=0 \Rightarrow$$

$$\hbar^2\left(\underbrace{\lambda-m_j^{max\,2}-m_j^{max}}_{=0}\right)|j,m_j^{max}>=0$$

$$\underline{\lambda=m_j^{max}(m_j^{max}+1)}$$

Similarly, $J_-|j,m_j^{min}>=0 \Rightarrow$

$$J_+\left(J_-|j,m_j^{min}>\right)=0 \Rightarrow$$

$$\left(\vec{J}^2-J_z^2+\hbar J_z\right)|j,m_j^{min}>=0 \Rightarrow$$

$$\hbar^2\left(\lambda-m_j^{min\,2}+m_j^{min}\right)|j,m_j^{min}>=0 \Rightarrow$$

$$\underline{\lambda=m_j^{min}(m_j^{min}-1)}$$

Since $m_j^{max}(m_j^{max}+1)=m_j^{min}(m_j^{min}-1) \Rightarrow$

$$\underline{m_j^{max}=-m_j^{min}}$$

To get from $|j,m_j^{max}>$ to $|j,m_j^{min}>$ by

... J_- $\Rightarrow$ takes $2m_j^{max} = k$ steps

$$k = 0, 1, 2, \ldots$$

Then, $\lambda = m_j^{max}(m_j^{max}+1) = \frac{k}{2}\left(\frac{k}{2}+1\right)$

Note that $m_j^{max} = \frac{k}{2} = 0, \frac{1}{2}, 1, \ldots$

$j = m_j^{max}$ ← can be integer or half-integer

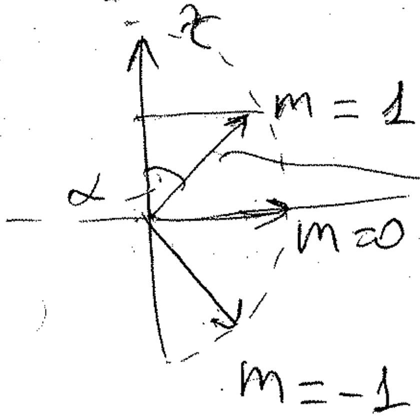
If $j \equiv l$ $\Rightarrow$ m_l is integer $\Rightarrow$ l is integer, $\lambda = l(l+1)$

↑ orbital angular momentum

If $j \equiv s$ $\Rightarrow$ m_s can be half-integer (e.g. for $s = \frac{1}{2}$ particles $m_s = \pm\frac{1}{2}$)

↑ spin angular momentum

Geometrical interpretation of $|l, m\rangle$



Example: $l = 1$, $m = -1, 0, 1$

$|\vec{L}| = \hbar\sqrt{l(l+1)}$

<u>note</u>: $|L_z| < |\vec{L}|$ $\Rightarrow$

$$\Updownarrow$$

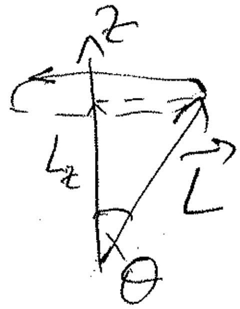
$$m^2 < l(l+1)$$

So, there is a minimum angle between $\vec{L}$ and z-axis $\Rightarrow \cos\alpha = \frac{L_z}{|\vec{L}|} = \frac{m_{max}}{\sqrt{\ell(\ell+1)}} = \frac{\ell}{\sqrt{\ell(\ell+1)}}$!

This is a consequence of the uncertainty principle

if $\vec{L}$ were parallel to z-axis $\Rightarrow$
$L_x = L_y = 0$, $L_z = \pm\hbar\ell$ $\Rightarrow$ we would know all three components of $\vec{L}$ simultaneously
Which is impossible, since $[L_i, L_j] \neq 0$! $(i \neq j)$

Instead, if we know $\vec{L}^2$ and L_z $\Rightarrow$ we can't know L_x and L_y with absolute precision $\Rightarrow$ geometrical interpretation is that $\vec{L}$ precesses around z-axis at a constant angle θ, which keeps L_z constant, but has L_x, L_y oscillating $\Rightarrow \langle L_x \rangle = \langle L_y \rangle = 0$, but $\langle L_x^2 \rangle = \langle L_y^2 \rangle \neq 0$!



Show this in HW #3 (Sakurai 3.16)