Lecture #15

Rotation matrices for coupling two angular momenta

Recall $\mathcal{D}(R) = \mathcal{D}_z(\alpha)\mathcal{D}_y(\beta)\mathcal{D}_z(\gamma)$

$$\mathcal{D}^{(j)}_{m'm} = \langle j, m' | \mathcal{D}^{(j)}(R) | j, m\rangle = e^{-i(m'\alpha + m\gamma)} \cdot \underbrace{\langle j, m' | e^{-\frac{i}{\hbar}\beta \hat{J}_y} | j, m\rangle}_{= d^{(j)}_{m'm}(\beta)}$$

If one knows $d^{(j_1)}_{m_1' m_1}(\beta)$ and $d^{(j_2)}_{m_2' m_2}(\beta)$, how does one calculate $d^{(j)}_{m'm}(\beta)$?

$$d^{(j)}_{m'm}(\beta) = \sum_{m_1', m_2', m_1, m_2} \langle j_1, j_2; m_1, m_2 | j, m\rangle \langle j_1, j_2; m_1', m_2' | j, m'\rangle \langle j_1, j_2; m_1', m_2' | e^{-\frac{i}{\hbar}\beta \hat{J}_y} | j_1, j_2; m_1, m_2\rangle$$

Since $J_y = J_{1y} + J_{2y}$ and $|j_1, j_2; m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \Rightarrow$

(2)

$$d^{(j)}_{m'm}(\beta) = \sum_{m_1,m_2} \sum_{m_1',m_2'} \langle j_1,j_2; m_1,m_2 | j,m\rangle \cdot$$

$$\cdot \langle j_1,j_2; m_1' m_2' | j, m'\rangle \underbrace{\langle j_1, m_1' | e^{-\frac{i}{\hbar}\beta J_{1y}} | j_1, m_1\rangle}_{"d^{(j_1)}_{m_1'm_1}(\beta)}$$

$$\cdot \underbrace{\langle j_2, m_2' | e^{-\frac{i}{\hbar}\beta J_{2y}} | j_2, m_2\rangle}_{"d^{(j_2)}_{m_2'm_2}(\beta)} =$$

$$= \sum_{m_1,m_2} \sum_{m_1',m_2'} \langle j_1,j_2; m_1,m_2 | j,m\rangle \cdot \langle j_1,j_2; m_1' m_2' | j m'\rangle$$

$$\cdot d^{(j_1)}_{m_1'm_1}(\beta)\, d^{(j_2)}_{m_2'm_2}(\beta)$$

Or (show)!

$$d^{(j_1)}_{m_1'm_1}(\beta)\, d^{(j_2)}_{m_2'm_2}(\beta) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m,m'} \langle j_1,j_2; m_1,m_2 | j,m\rangle$$

$$\cdot \langle j_1,j_2; m_1', m_2' | j, m'\rangle\, d^{(j)}_{m'm}(\beta)$$

Similarly, $\longleftrightarrow$ Clebsch-Gordan series

$$\mathcal{D}^{(j_1)}_{m_1'm_1}(\alpha,\beta,\gamma)\, \mathcal{D}^{(j_2)}_{m_2'm_2}(\alpha,\beta,\gamma) = \sum_{j,m,m'} \langle j_1,j_2; m_1,m_2 | j, m\rangle$$

$$\cdot \langle j_1,j_2; m_1', m_2' | j, m'\rangle\, \mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma) \qquad (17.1)$$

One of the applications of (17.1): $m_1 = m_2 = 0$ (3)

$j_1 = l_1$, $j_2 = l_2$ and $m = 0$ $\Rightarrow$

$$\mathcal{D}^{(l_1)}_{m_1' 0}(\alpha,\beta,\gamma)\,\mathcal{D}^{(l_2)}_{m_2' 0}(\alpha,\beta,\gamma) = \sum_{l,m'} \langle l_1, l_2; 0,0 | l, 0\rangle \cdot \langle l_1, l_2; m_1' m_2' | l, m'\rangle\, \underbrace{\mathcal{D}^{(l)}_{m' 0}(\alpha,\beta,\gamma)}_{\sqrt{\frac{4\pi}{2l+1}}\, Y_l^{m'}(\theta,\varphi)}$$

Lecture #13

$$Y_{l_1}^{m_1'}(\theta,\varphi)\, Y_{l_2}^{m_2'}(\theta,\varphi) = \sum_{l,m'} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} \cdot \langle l_1, l_2; 0,0 | l,0\rangle \langle l_1, l_2; m_1, m_2 | l, m'\rangle Y_l^{m'}(\theta,$$

Multiply by $Y_l^{m'*}$ and integrate;

$$\int Y_l^{m'*}(\theta,\varphi)\, Y_{l_1}^{m_1}(\theta,\varphi)\, Y_{l_2}^{m_2}(\theta,\varphi)\, d\Omega =$$

$$= \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\, \langle l_1, l_2; 0,0 | l, 0\rangle \langle l_1, l_2; m_1, m_2 | l, m'$$

from tables of C.-G. coeff.

useful for calculating probabilities of transitions in spectroscopy

Spectroscopic notations

(a) Capital letters $S, P, D \ldots \to$ refer to the values of orbital angular momentum
$l = 0 \quad 1 \quad 2 \ldots$

(b) Subscript on the right $\Rightarrow j$

(c) Superscript on the left $\Rightarrow 2S+1$

Example $\quad {}^2P_{3/2} \Rightarrow \begin{array}{l} l = 1 \\ S = 1/2,\ j = 3/2 \end{array}$

Example problem. A hydrogen atom is in a ${}^2P_{1/2}$ state with total angular momentum up along z-axis. With what probability will the electron be found with spin down?

Solution: $\quad {}^2P_{1/2} \Rightarrow \begin{array}{l} l = 1 \quad (= j_1) \\ s = \frac{1}{2} \quad (= j_2) \\ j = \frac{1}{2} \ ; \ m = \frac{1}{2} \end{array}$

Need to find probability that the electron is in the $m_s = -\frac{1}{2}$ state $\Rightarrow$ transform coupled representation into uncoupled $\Rightarrow$

$$\underbrace{|\tfrac{1}{2}, \tfrac{1}{2}>}_{\substack{j \quad m \\ j_1+j_2-1 = l+s-1}} \underset{\text{Eq.(16.3)}}{=} -\sqrt{\frac{1/2}{1/2+1}}\ |\underset{\substack{j_1 \\ l}}{1}, \underset{\substack{j_2 \\ s}}{\tfrac{1}{2}} ; \underset{\substack{j_1-1 \\ m_l}}{0}, \underset{\substack{j_2 \\ m_s}}{\tfrac{1}{2}}> + \sqrt{\frac{1}{1/2+1}}\ |\underset{l}{1}, \underset{s}{\tfrac{1}{2}}, \underset{m_l}{1}, \underset{m_s}{\tfrac{1}{2}}$$

$$= -\sqrt{\frac{1}{3}}\ |1, \tfrac{1}{2} ; 0, \tfrac{1}{2}> + \sqrt{\frac{2}{3}}\ |1, \tfrac{1}{2} ; 1, -\tfrac{1}{2}> \quad (17.2)$$

The state corresponding to the "spin down", i.e.

$m_s = -\frac{1}{2}$ is $\underbrace{|1, \tfrac{1}{2}; 1, -\tfrac{1}{2}\rangle}_{l\ \ s\ \ m_l\ \ m_s} \Rightarrow$

$$\mathcal{P}_{m_s = -\frac{1}{2}} = \frac{2}{3}$$

Note: How would we rewrite Eq. (19.2) in terms of a two-component spinor in this example problem above?

So,
$$\underbrace{\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle}_{j\ \ m} = \sqrt{\tfrac{2}{3}}\,\underbrace{\left|1, \tfrac{1}{2}; 1, -\tfrac{1}{2}\right\rangle}_{l\ \ s\ \ m_l\ \ m_s} - \sqrt{\tfrac{1}{3}}\,\underbrace{\left|1, \tfrac{1}{2}; 0, \tfrac{1}{2}\right\rangle}_{l\ \ s\ \ m_l\ \ m_s}$$

$$\Psi_+ \Leftarrow Y_1^0 \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

$$\Downarrow$$

$$Y_1^1(\theta, \varphi) \otimes \underbrace{\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle}_{s\ \ m_s} \Rightarrow \Psi_-$$

$$\Downarrow$$

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \doteq \frac{1}{\sqrt{3}} \begin{bmatrix} -Y_1^0 \\ \sqrt{2}\, Y_1^1 \end{bmatrix}$$

Examples of using a table of Clebsch–Gordan coeff.
(attached)

for $j_1 = j_2 = 1 \Rightarrow$

$$\underbrace{|2, 1\rangle}_{J\ \ M} = \frac{1}{\sqrt{2}}\,\underbrace{|1, 1; 1, 0\rangle}_{j_1\ j_2\ m_1\ m_2} + \frac{1}{\sqrt{2}}\,\underbrace{|1, 1; 0, 1\rangle}_{j_1\ j_2\ m_1\ m_2}$$

$$|2, 0\rangle = \frac{1}{\sqrt{6}}\,|1, 1; 1, -1\rangle + \sqrt{\tfrac{2}{3}}\,|1, 1; 0, 0\rangle + \frac{1}{\sqrt{6}}\,|1, 1; -1, 1\rangle$$