# Bell's theorem: experimental tests and implications

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## Abstract

Bell's theorem represents a significant advance in understanding the conceptual foundations of quantum mechanics. The theorem shows that essentially all local theories of natural phenomena that are formulated within the framework of realism may be tested using a single experimental arrangement. Moreover, the predictions by these theories must significantly differ from those by quantum mechanics. Experimental results evidently refute the theorem's predictions for these theories and favour those of quantum mechanics. The conclusions are philosophically startling: either one must totally abandon the realistic philosophy of most working scientists, or dramatically revise our concept of space–time.

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## 1. Introduction

Realism is a philosophical view, according to which external reality is assumed to exist and have definite properties, whether or not they are observed by someone. So entrenched is this viewpoint in modern thinking that many scientists and philosophers have sought to devise conceptual foundations for quantum mechanics that are clearly consistent with it. One possibility, it has been hoped, is to reinterpret quantum mechanics in terms of a statistical account of an underlying hidden-variables theory in order to bring it within the general framework of classical physics. However, Bell's theorem has recently shown that this cannot be done. The theorem proves that all realistic theories, satisfying a very simple and natural condition called locality, may be tested with a single experiment against quantum mechanics. These two alternatives necessarily lead to significantly different predictions. The theorem has thus inspired various experiments, most of which have yielded results in excellent agreement with quantum mechanics, but in disagreement with the family of local realistic theories. Consequently, it can now be asserted with reasonable confidence that either the thesis of realism or that of locality must be abandoned. Either choice will drastically change our concepts of reality and of space–time.

The historical background for this result is interesting, and represents an extreme irony for Einstein's steadfastly realistic position, coupled with his desire that physics be expressable solely in simple geometric terms. Within the realistic framework, Einstein *et al* (1935, hereafter referred to as EPR) presented a classic argument. As a starting point, they assumed the non-existence of action-at-a-distance and that some of the statistical predictions of quantum mechanics are correct. They considered a system consisting of two spatially separated but quantum-mechanically correlated particles. For this system, they showed that the results of various experiments are predetermined, but that this fact is not part of the quantum-mechanical description of the associated systems. Hence that description is an incomplete one. To complete the description, it is thus necessary to postulate additional 'hidden variables', which presumably will then restore completeness, determinism and causality to the theory.

Many in the physics community rejected their argument, preferring to follow a counter-argument by Bohr (1935), who believed that the whole realistic viewpoint is inapplicable. Many others, however, felt that since both viewpoints lead to the same observable phenomenology, a commitment to either one is only a matter of taste. Hence, the discussion, for the greater part of the subsequent 30 years, was pursued perhaps more at physicists' cocktail parties than in the mainstream of modern research.

Starting in 1965, however, the situation changed dramatically. Using essentially the same postulates as those of EPR, J S Bell showed for a *Gedankenexperiment* of Bohm (a variant of that of EPR) that no deterministic local hidden-variables theory can reproduce all of the statistical predictions by quantum mechanics. Inspired by that work, Clauser *et al* (1969, hereafter referred to as CHSH) added three contributions. First, they showed that his analysis can be extended to cover actual systems, and that experimental tests of this broad class of theories can be performed. Second, they introduced a very reasonable auxiliary assumption which allows tests to be performed

with existing technology. Third, they specifically proposed performing such a test by examining the polarisations of photons produced by an atomic cascade, and derived the required conditions for such an experiment.

Curiously, the transition to a consideration of real systems introduced new aspects to the problem. EPR had demonstrated that any ideal system which satisfies a locality condition must be deterministic (at least with respect to the correlated properties). Since that argument applies only to ideal systems, CHSH therefore had postulated determinism explicitly. Yet, it eventually became clear that it is not the deterministic character of these theories that is incompatible with quantum mechanics. Although not stressed, this point was contained in Bell's subsequent papers (1971, 1972)—any non-deterministic (stochastic) theory satisfying a more general locality condition is also incompatible with quantum mechanics. Indeed it is the objectivity of the associated systems and their locality which produces the incompatibility. Thus, the whole realistic philosophy is in question! Bell's (1971) result, however, is in a form that is awkward for an experimental test. To facilitate such tests, Clauser and Horne (1974, hereafter referred to as CH) explicitly characterised this broad class of theories. They then gave a new incompatibility theorem that yields an experimentally testable result and derived the requirements for such a test. Although such an experiment is difficult to perform (and in fact has not yet been performed), they showed that an assumption weaker in certain respects than the one of CHSH allowed the experiments proposed earlier by CHSH to be used as a test for these theories also.

The interpretation of all of the existing results requires at least some auxiliary assumptions, although experiments are possible for which this is not the case. Even though some of the assumptions are very reasonable, this fact allows loopholes still to exist. Experiments now in progress or being planned will be able to eliminate most of these loopholes. However, even now one can assert with reasonable confidence that the experimental evidence to date is contrary to the family of local realistic theories. The construction of a quantum-mechanical world view as an alternative to the point of view of the local realistic theories is beyond the scope of this review.

Section 2 of this review summarises the argument of EPR, appendix 1 discusses various critical evaluations of it, and appendix 2 summarises briefly the history of hidden-variables theories. Section 3 describes the versions of Bell's theorem discussed above as well as some others. Section 4 discusses the requirements for a fully general test and shows why such an experiment is a difficult one to perform. Section 5 is devoted to a description of the cascade-photon experiments proposed by CHSH. First, it discusses the auxiliary assumptions by CHSH and CH. Second, calculations of the quantum-mechanical predictions for these experiments are summarised. Third, there is a discussion of the actual cascade-photon experiments performed so far (Freedman and Clauser 1972, Holt and Pipkin 1973, Clauser 1976, Fry and Thompson 1976). All but the second agree very well with the quantum-mechanical predictions, thus providing significant evidence against the entire family of local realistic theories. Section 5 ends with a critique of the CH and CHSH assumptions. Section 6 summarises and discusses related experiments measuring the polarisation correlation of photons produced in positronium annihilation (Kasday *et al* 1975, Faraci *et al* 1974, Wilson *et al* 1976, Bruno *et al* 1977) and an experiment measuring the spin correlation of proton pairs (Lamehi-Rachti and Mittig 1976). Section 7 is devoted to an evaluation of the experimental results obtained so far and to the prospects for future experiments.

## 2. The Einstein–Podolsky–Rosen argument

A profound argument for the thesis that a quantum-mechanical description of a physical system is incomplete was presented by EPR in 1935. Their argument rests upon three premises. (i) Some of the quantum-mechanical predictions concerning observations on a certain type of system, consisting of two spatially separated particles, are correct. (ii) A very reasonable criterion for the existence of 'an element of physical reality' is proposed: 'If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity' (EPR 1935, p777). (iii) There is no action-at-a-distance in nature.

The system which they study consists of two particles, which are prepared in a state such that the sum of their momenta in a given direction $(p_1+p_2)$ and the difference of their positions $(x_1-x_2)$ are both definite. The wavefunction $\delta(x_1-x_2-a)$ quantum mechanically describes this system, for it is an eigenfunction of the operator $x_1-x_2$ with eigenvalue $a$, and of the operator $p_1+p_2$ with eigenvalue 0. By measuring the position of particle 1 one can predict with certainty, according to quantum mechanics, what value will be found if the position of particle 2 is then measured (immediately). In view of premise (iii) the prediction is made without in any way disturbing particle 2, since the two particles are spatially separated. EPR therefore infer that the position of particle 2 has a definite predetermined value, not included in the description by the wavefunction $\delta(x_1-x_2-a)$. By an analogous argument EPR also infer that the momentum of particle 2 has a definite value, contrary to the uncertainty principle. (Of course, the same argument, starting with measurements made upon particle 2, allows them to infer that particle 1 also has both a definite position and a definite momentum.) Hence EPR reach the conclusion that at least in this particular situation the quantum-mechanical description is incomplete. Although they do not use the term 'hidden variables', this expression can be appropriately used to apply to the parts of the complete state which are not comprised in the quantum-mechanical description, and which suffice to fix the outcomes of measurements that are not fully determined quantum mechanically.

In our opinion the reasoning of EPR is impeccable, once an ambiguity in the phrase 'can predict', which occurs in the second premise, is removed. In a narrow sense, one can predict the value of a quantity only *when an experimental arrangement is chosen for determining the value of that quantity*. In a broad sense, one can predict the value of a quantity *if it is possible to choose an experimental arrangement for determining it*. If the narrow sense is accepted, then the argument of EPR clearly does not go through, since the experimental arrangements for measuring the position and momentum of a particle are incompatible. From the standpoint of physical realism the broad sense of 'can predict' is the appropriate one, since from that viewpoint, one conceives a physical system to have a definite set of properties independently of their being observed, but which may of course be explored at the option of the experimenter. In the situation envisaged by EPR one can predict, in the broad sense, both $x_2$ and $p_2$. Hence if this sense of the ambiguous phrase is adopted, their argument does go through. An assumption of physical realism clearly underlies the argument by EPR. Bohr's (1935) answer to EPR, defending the completeness of quantum mechanics, consisted essentially of a critique of the realism which they had taken for granted (see appendix 1).

A variant of EPR's argument was given by Bohm (1951), formulated in terms of discrete states. He considered a pair of spatially separated spin-$\frac{1}{2}$ particles produced

somehow in a singlet state, for example, by dissociation of the spin-0 system. Various spin components of each of these particles may then be measured independently at the option of the experimenter. The spin part of the state vector is given by:

$$\Psi = \frac{1}{\sqrt{2}} \left[ u_{\hat{n}}^+(1) \otimes u_{\hat{n}}^-(2) - u_{\hat{n}}^-(1) \otimes u_{\hat{n}}^+(2) \right].  \tag{2.1}$$

Here $\boldsymbol{\sigma} \cdot \hat{n} u_{\hat{n}}^{\pm}(1) = \pm u_{\hat{n}}^{\pm}(1)$, so that $u_{\hat{n}}^{\pm}(1)$ quantum mechanically describes a state in which particle 1 has spin 'up' or 'down', respectively, along the direction $\hat{n}$; $u_{\hat{n}}^{\pm}(2)$ has an analogous meaning concerning particle 2. Since the singlet state $\Psi$ is spherically symmetric, $\hat{n}$ can specify any direction. Suppose that one measures the spin of particle 1 along the $\hat{x}$ axis. The outcome is not predetermined by the description $\Psi$. But from it, one can predict that if particle 1 is found to have its spin parallel to the $\hat{x}$ axis, then particle 2 will be found to have its spin antiparallel to the $\hat{x}$ axis if the $\hat{x}$ component of its spin is also measured. Thus, the experimenter can arrange his apparatus in such a way that he can predict the value of the $\hat{x}$ component of spin of particle 2 presumably without interacting with it (if there is no action-at-a-distance). Likewise, he can arrange the apparatus so that he can predict any other component of the spin of particle 2. The conclusion of the argument is that all components of spin of each particle are definite, which of course is not so in the quantum-mechanical description. Hence, a hidden-variables theory seems to be required.

Some comments are in order concerning EPR's premises in the light of Bell's theorem. If premise (i) is taken to assert that all of the quantum-mechanical predictions are correct, then Bell's theorem has shown it to be inconsistent with premises (ii) and (iii). Actually, in the body of their argument EPR used only a few predictions with probability one, which are atypical in quantum mechanics, whereas the discrepancies which Bell exhibited between local realistic theories and quantum mechanics involved statistical predictions. If it was EPR's intention to aim at a hidden-variables theory which is local and realistic, and which agrees with all the statistical predictions of quantum mechanics—as many readers have understood them—then, of course, Bell's theorem shows mathematically that this aim cannot be achieved. We shall not try to answer the historical question of their intent. Two statements, however, can be made with confidence. First, the argument from their premises is valid, once the above-mentioned ambiguity is cleared up. Second, the physical situation which they envisaged is of immense value for examining the philosophical implications of quantum mechanics and (via Bell's work) for exploring the limitations of the family of local realistic physical theories.

## 3. Bell's theorem

There is a vast literature concerning the consistency of hidden-variables theories with the algebraic structure of the observables of quantum mechanics. The major results of this literature are summarised in appendix 2, but they are not indispensable for understanding the content and implications of Bell's theorem. Heuristically, however, this literature was very important for Bell's work. In the course of preparing a review article on 'impossibility' proofs of hidden-variables interpretations of quantum mechanics, Bell studied the theories proposed by de Broglie (1928) and Bohm (1952). He noticed, as Bohm had already realised, that in order to reproduce the quantum-theoretic predictions for a system of EPR type, they postulated the

existence of non-local interactions between spatially separated particles. Bell was thus led to ask whether the peculiar non-locality exhibited by these models is a generic characteristic of hidden-variables theories that agree with the statistical predictions by quantum mechanics. He proved (Bell 1965) that the answer is positive for the whole class of deterministic hidden-variables theories in the domain of ideal apparatus and systems. Stronger versions of this theorem, which also constrain actual systems, were later proved by Bell himself and by others. These versions state that essentially all realistic local theories of natural phenomena may be tested in a single experimental arrangement against quantum mechanics, and that these two alternatives necessarily lead to observably different predictions.

In this section we review some of these derivations, which we shall refer to collectively as 'Bell's theorem'. Our purpose here is to arrive at versions of Bell's theorem which satisfy the following criteria. (i) The hypotheses seem to be inescapable for anyone who is committed to physical realism and to the non-existence of action-at-a-distance. (ii) Discrepancies with the predictions by quantum mechanics occur in at least one situation which is experimentally realisable. Criterion (i) is, in our opinion, very close to having been achieved, although the hypotheses are violated by some pathological instances of local realistic theories. Criterion (ii) has essentially been achieved; however, the experiment which it specifies is difficult, and has not yet been performed. Additional assumptions, not implicit in locality and realism, have been relied upon to allow easier experiments to be considered. (The assumptions and experiments are discussed in §§5 and 6.) Unfortunately, this fact leaves open various loopholes (discussed in §§5–7). It must be stressed, however, that the existence of these loopholes in no way diminishes the mathematical validity of the versions of Bell's theorem presented in this section.

### 3.1. *Deterministic local hidden-variables theories and Bell (1965)*

In his paper of 1965 Bell considered Bohm's *Gedankenexperiment*, described above in §2. That system consists of two spin-$\frac{1}{2}$ particles, prepared in the quantum-mechanical singlet state $\Psi$ given by equation (2.1). Let $A_{\hat{a}}$ be the result of a measurement of the spin component of particle 1 of the pair along the direction $\hat{a}$, and let $B_{\hat{b}}$ be that of particle 2 along direction $\hat{b}$. We take the unit of spin as $\hbar/2$; hence, $A_{\hat{a}}, B_{\hat{b}} = \pm 1$.

The product $A_{\hat{a}}.B_{\hat{b}}$ is a single observable of the two-particle system (even though two distinct operations are needed in order to measure it). It is represented quantum mechanically by a self-adjoint operator on the Hilbert space associated with the system. For this *Gedankenexperiment* one can readily calculate the quantum-mechanical prediction for the expectation value of this observable[†]:

$$[E(\hat{a}, \hat{b})]_{\Psi} = \langle \Psi | \boldsymbol{\sigma}_1.\hat{a}\boldsymbol{\sigma}_2.\hat{b} | \Psi \rangle = -\hat{a}.\hat{b}. \tag{3.1}$$

A special case of equation (3.1) contains the determinism implicit in this idealised system. When the analysers are parallel, we have:

$$[E(\hat{a}, \hat{a})]_{\Psi} = -1 \tag{3.2}$$

for all $\hat{a}$. Thus, one can predict with certainty the result $B$, by previously obtaining

[†] The notation of this review is to use the wavefunction or the letters QM as a subscript to denote the quantum-mechanical prediction. We omit the subscript for predictions by the class of theories included by the postulates of Bell's theorem, when this convention does not cause confusion.

the result $A$ (EPR's premise (ii)). Since the quantum-mechanical state $\Psi$ does not determine the result of an individual measurement, this fact (via EPR's argument) suggests that there exists a more complete specification of the state in which this determinism is manifest. We denote this state by the single symbol $\lambda$, although it may well have many dimensions, discrete and/or continuous parts, and different parts of it interacting with either apparatus, etc. Presumably the quantum state $\Psi$ is a related partial specification of this state. We thus define a deterministic hidden-variables theory as any physical theory which postulates the existence of states of a system, for which the observables of quantum mechanics always have definite values.

Let $\Lambda$ be the space of the states $\lambda$ for an ensemble comprised of a very large number of the observed systems. We make no restrictions as to what type of space this is, nor to its dimensionality, nor do we require linearity for operations with it, but of course we require that a set of Borel subsets of $\Lambda$ be defined, so that probability measures can be defined upon it. We represent the distribution function for the states $\lambda$ on the space $\Lambda$ by the symbol $\rho$. For this ensemble we take $\rho$ to have norm one:

$$\int_\Lambda \mathrm{d}\rho = 1. \tag{3.3}$$

In a deterministic hidden-variables theory the observable $A_{\hat{a}}.B_{\hat{b}}$ has a definite value $(A_{\hat{a}}.B_{\hat{b}})(\lambda)$ for the state $\lambda$. For these theories Bell defined locality as follows: *a deterministic hidden-variables theory is local if for all $\hat{a}$ and $\hat{b}$ and all $\lambda \in \Lambda$ we have:*

$$(A_{\hat{a}}.B_{\hat{b}})(\lambda) = A_{\hat{a}}(\lambda).B_{\hat{b}}(\lambda). \tag{3.4}$$

That is, once the state $\lambda$ is specified and the particles have separated, measurements of $A$ can depend only upon $\lambda$ and $\hat{a}$ but not $\hat{b}$. Likewise measurements of $B$ depend only upon $\lambda$ and $\hat{b}$. Any reasonable physical theory that is realistic and deterministic and that denies the existence of action-at-a-distance is local in this sense. (More general definitions of 'local' will be considered in §3.3.) For such theories the expectation value of $A_{\hat{a}}.B_{\hat{b}}$ is then given by

$$E(\boldsymbol{a}, \boldsymbol{b}) = \int_\Lambda A_{\hat{a}}(\lambda) B_{\hat{b}}(\lambda)\, \mathrm{d}\rho. \tag{3.5}$$

Bell's (1965) proof of the theorem consists of showing that if the locality condition (3.4) and the condition (3.2) for partial agreement with quantum mechanics are both satisfied, then the expectation values satisfy a simple inequality. This inequality is then an alternative prediction to that by quantum mechanics for the expectation value of $A_{\hat{a}}.B_{\hat{b}}$. The predictions made by this inequality are quantitatively different from those of equation (3.1).

The demonstration is straightforward. Equation (3.2) can hold if and only if

$$A_{\hat{a}}(\lambda) = -B_{\hat{a}}(\lambda) \tag{3.6}$$

holds for all $\lambda \in \Lambda$. Using equation (3.6) we calculate the following function, which involves three different possible orientations of the analysers:

$$E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) = -\int_\Lambda [A_{\hat{a}}(\lambda) A_{\hat{b}}(\lambda) - A_{\hat{a}}(\lambda) A_{\hat{c}}(\lambda)]\, \mathrm{d}\rho$$

$$= -\int_\Lambda A_{\hat{a}}(\lambda) A_{\hat{b}}(\lambda)[1 - A_{\hat{b}}(\lambda) A_{\hat{c}}(\lambda)]\, \mathrm{d}\rho.$$

Since $A, B = \pm 1$, this last expression can be written:

$$|E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c})| \leqslant \int_\Lambda [1 - A_{\hat{b}}(\lambda) A_{\hat{c}}(\lambda)]\, \mathrm{d}\rho.$$

Using equations (3.3), (3.5) and (3.6) we have:

$$|E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c})| \leqslant 1 + E(\hat{b}, \hat{c}). \tag{3.7}$$

Inequality (3.7) is the first of a family of inequalities which are collectively called 'Bell's inequalities'.

A simple instance of the disagreement between the predictions of equation (3.1) and inequality (3.7) is provided by taking $\hat{a}$, $\hat{b}$ and $\hat{c}$ to be coplanar, with $\hat{c}$ making an angle of $2\pi/3$ with $\hat{a}$, and $\hat{b}$ making an angle of $\pi/3$ with both $\hat{a}$ and $\hat{c}$. Then:

$$\hat{a}.\hat{b} = \hat{b}.\hat{c} = \tfrac{1}{2} \qquad \hat{a}.\hat{c} = -\tfrac{1}{2}.$$

For these directions:

$$|[E(\hat{a}, \hat{b})]_\Psi - [E(\hat{a}, \hat{c})]_\Psi| = 1 \qquad \text{while} \qquad 1 + [E(\hat{b}, \hat{c})]_\Psi = \tfrac{1}{2}.$$

These values do not satisfy inequality (3.7). Hence the quantum-mechanical prediction and that by inequality (3.7) are incompatible, at least for some pairs of analyser orientations.

The version of Bell's theorem just proved can be summarised as follows: no deterministic hidden-variables theory satisfying equation (3.2) and the locality condition (3.1) can agree with all of the predictions by quantum mechanics concerning the spins of a pair of spin-$\tfrac{1}{2}$ particles in the singlet state.

### 3.2. Foreword to the non-idealised case

Any argument whose scope is strictly limited to a discussion of ideal systems is of little value to working physicists, who endeavour to describe systems that can and do occur in practice. The immense heuristic value of Bell's (1965) argument, outlined in §3.1, is that it leads to formulations that provide direct experimental predictions for systems which can actually be produced in a laboratory. By itself, the derivation given in §3.1 is insufficient to do this, because of its reliance upon the existence of a pair of analyser orientations for which there is a perfect correlation. That is, the above proof hinges strongly upon the condition that equation (3.2) hold exactly. Use is made of this equation in three ways. First, it allows the proof to go through mathematically. Second, determinism is derivable from it and does not have to be postulated separately. Finally, for reasons to be discussed, it assumes that the locality postulate is reasonable.

Unfortunately, equation (3.2) cannot hold exactly in an actual experiment. Any real detector will have an efficiency less than 100%, and any real analyser will have some attenuation as well as some leakage into its orthogonal channel. Since we are attempting to deal with not just one but a whole class of theories, it is quite possible that in some of these theories the above imperfections are inherently correlated with the measurement and detection processes in a way that depends upon the state $\lambda$. The problems which arise when these three implications cannot be drawn will be considered in turn.

The problem concerning the derivation's mathematical reliance upon equation (3.2) was first solved by CHSH. They demonstrated that a different proof of the theorem follows from the above formalism, without requiring equation (3.2) to hold. They derived a different inequality that is violated by the quantum-mechanical predictions for systems which never achieve the perfect correlation of equation (3.2), but which do achieve a necessary minimum correlation. The inequality which results

from their analysis is:

$$|E(a, b) - E(a', b)| + E(a, b') + E(a', b') \leqslant 2. \qquad (3.8)$$

When equation (3.2) does hold, inequality (3.8) implies inequality (3.7) as a special case. Since essentially this same inequality was subsequently derived by Bell (1971) for the more general non-deterministic case presented in §3.4, we will not present the CHSH derivation here.

The second problem—that determinism is no longer derivable—is not a serious one. One needs merely to assume that determinism holds for the theories under consideration. Indeed this was the approach by CHSH. Thereby, they produced a very powerful result, which constrains deterministic local hidden-variables theories for realisable systems. However, it was subsequently noticed by Bell (1971, 1972) and Clauser and Horne (1974) that this assumption is not needed. On the contrary, a weakening of the locality requirement can be made which still allows inequality (3.8) to be derived, but which significantly increases the scope of the theorem. The theorem thus applies to a class of fundamentally stochastic theories, as well as to deterministic theories in which there are hidden variables in the apparatus.

The third problem is a very delicate one, yet one of great importance. In the idealised situation, whenever a particle is observed at one apparatus an associated particle is *always* observed at the other apparatus. The selection of the sub-ensemble of observed particles from among all of those emitted by the source depends only upon the collimator and source geometry and can have no dependence upon the parameters $\hat{a}$ and $\hat{b}$. Hence $\rho$ was defined for the observed particles, and one can then be confident that it is independent of $\hat{a}$ and $\hat{b}$.

In the actual case, on the other hand, observed particles are paired with particles which, for some reason, are not observed at all, i.e. in neither a spin-up nor a spin-down channel.

The sub-ensemble which we used in the idealised case is then further partitioned into four disjoint sub-ensembles, i.e. those for which (*a*) both particles are observed, (*b*) only particle 1 is observed, (*c*) only particle 2 is observed, and (*d*) neither particle is observed. The distribution $\rho$ of the union of these four sets is clearly independent of $\hat{a}$ and $\hat{b}$. However, the mode of partitioning may well depend upon $\hat{a}$ and $\hat{b}$, since the detection and various attenuation processes occur 'downstream' from the analysers. Hence there is no reason to expect that the composition, and thus the distribution, of each sub-ensemble is independent of $\hat{a}$ and $\hat{b}$. (This fact was noticed by Pearle (1970) and Clauser and Horne (1974). The latter contrived a hidden-variables theory in which $\rho$ becomes dependent upon $\hat{a}$ and $\hat{b}$ when sub-ensemble (*a*) alone is considered and which yields exactly the quantum-mechanical predictions for the system.) Thus if we are to use equation (3.3) for a normalisation condition, and to expect that $\rho$ is independent of $\hat{a}$ and $\hat{b}$, the ensemble for which it is defined must also include the unobserved particles. Since their number is unknown and may be very large, it is no longer obvious how to compare the prediction by inequality (3.8) with experiment.

Three approaches to this problem have been pursued. The approach used by CHSH is to introduce an auxiliary assumption, that if a particle passes through a spin analyser, its probability of detection is independent of the analyser's orientation. Unfortunately, this assumption is not contained in the hypotheses of locality, realism or determinism. Moreover, it also has the undesirable feature that it makes the process of 'passage' or 'non-passage' a primitive one, and thereby excludes from consideration theories for which partial passage is appropriate.

A second approach, that used by Bell (1971) (although not specifically stated but clear from the context), is to employ an auxiliary apparatus ('event-ready' detectors) to measure the number of pairs emitted by the source. This possibility is shown schematically in figure 1. For this scheme, one can simply take the ensemble to consist of the particles which actually trigger the 'event-ready' detectors. Whether or not a triggering occurs clearly does not depend upon the analyser orientations. No problem with locality arises from the presence of the signals propagating to the remote apparatuses, since these signals can be simply considered as part of the state $\lambda$. Unfortunately, in practice most conceivable 'event-ready' detectors depolarise or destroy the particles. The value of this approach is thus limited.

An altogether different approach was employed by Clauser and Horne (1974). They derived an inequality from the hypotheses of locality and realism in which only ratios of the observed particle detection probabilities appear, and the normalisation condition equation (3.3) is not required for its derivation. The influence of the size
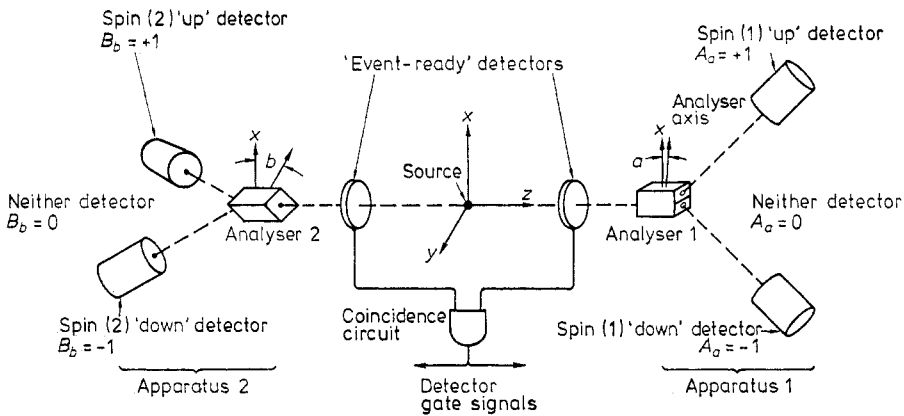


**Figure 1.** Apparatus configuration used for Bell's 1971 proof. 'Event-ready' detectors signal both arms that a pair of particles has been emitted. For a given gate signal, the result on either arm is assigned the value $+1$ if the corresponding spin-up detector responds, $-1$ if the spin-down detector responds, and 0 if neither detector responds.

of the ensemble thus vanishes. Their apparatus arrangement does not have the 'event-ready' detectors of figure 1, nor does it have two detectors for each apparatus but only one. It is thus much simpler, and is shown schematically in figure 2.

In the remainder of this section, we will show how these latter two approaches proceed. First, however, we will discuss the aforementioned generalisation of the locality postulate to include inherently stochastic theories.

### 3.3. Generalisation of the locality concept

Consider either of the experimental configurations for Bohm's *Gedankenexperiment*, described in §3.2. Actually there is nothing in the proof which requires the systems to be spin-$\frac{1}{2}$ particles. They may be any discrete-state quantum-mechanically correlated emissions. (However, not all quantum-mechanically correlated systems are predicted to violate the resulting inequalities. A careful choice is required to find one which is an appropriate test case.) In Bohm's *Gedankenexperiment* the symbols $\hat{a}$ and $\hat{b}$ are taken to represent the orientations of the Stern–Gerlach magnets used for
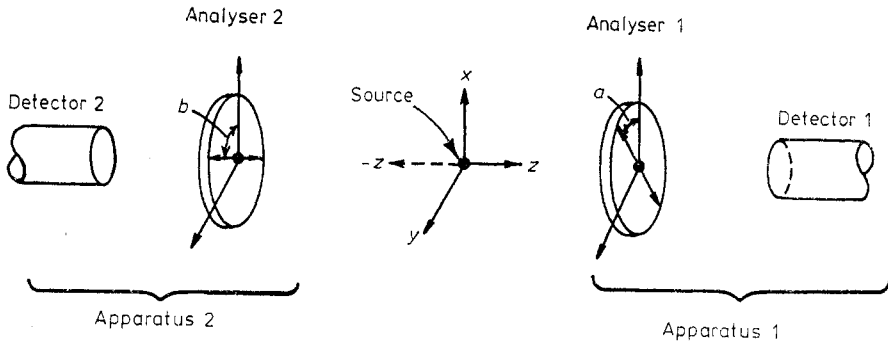
**Figure 2.** Apparatus configuration used in the proofs by CHSH and by CH. A source
emitting particle pairs is viewed by two apparatuses. Each apparatus consists of an
analyser and an associated detector. The analysers have parameters $a$ and $b$ res-
pectively, which are externally adjustable. In the above example $a$ and $b$ represent
the angles between the analyser axes and a fixed reference axis.

measuring the associated spin components. However, in general, $a$ and $b$ may rep-
resent any associated apparatus parameters under control by the experimenter. As
before, $A_a$ and $B_b$ represent the measurement outcomes at apparatuses 1 and 2,
respectively. Appropriate values will be assigned to these outcomes, as necessary.

The preceding definition of locality will now be generalised to include systems
whose evolution is inherently stochastic, as well as to include deterministic systems
with additional random variables associated with either apparatus, and that may locally
affect their experimental outcomes. Suppose a pair of correlated systems, which have
a joint state $\lambda$, separate. They then continue to evolve perhaps in an inherently
stochastic way, and given $\lambda$, $a$ and $b$, one can define probabilities for any particular
outcome at either apparatus. We allow that, given $\lambda$, these two probabilities may each
depend upon the associated (local) apparatus parameter, $a$ or $b$ respectively, and of
course upon $\lambda$, but we assume that these probabilities are otherwise independent of
each other.

This definition of locality seems very common-sensical. It says that the outcome
(or the probability of outcomes) of a measurement performed on one part of a compo-
site system is independent of what aspects of the other component the experimenter
chooses to measure. It by no means excludes the possibility of obtaining knowledge
concerning system 2 from an examination of system 1. The state $\lambda$ contains informa-
tion common to both systems, and a measurement on one of these presumably reveals
some of this. Nor does it prevent a measurement performed on one component of a
composite system from locally disturbing that component. What it does prescribe,
in essence, is that the measured value of a quantity on one system is not causally
affected by what one chooses to measure on the other system, since the two systems
are well separated (e.g. space-like separated) when the measurements are performed.

*3.4. Bell's 1971 proof*

We now describe Bell's (1971) proof, using this generalised locality definition.
The apparatus configuration appropriate to this proof was discussed in §3.2 and is
shown in figure 1. Given that a particle pair was emitted into the associated appara-
tuses, the results of either measurement can have one of three possible outcomes, to

which the following values were assigned by Bell:

$$A_a(\lambda) = \begin{cases} +1, & \text{'spin-up' detector triggered by particle 1} \\ -1, & \text{'spin-down' detector triggered by particle 1} \\ 0, & \text{particle 1 not detected} \end{cases} \qquad (3.9(a))$$

and

$$B_b(\lambda) = \begin{cases} +1, & \text{'spin-up' detector triggered by particle 2} \\ -1, & \text{'spin-down' detector triggered by particle 2} \\ 0, & \text{particle 2 not detected.} \end{cases} \qquad (3.9(b))$$

For a given state $\lambda$ of the emitted composite system, we denote the expectation values for these quantities by the symbols $\bar{A}_a(\lambda)$ and $\bar{B}_b(\lambda)$. In the general case these average values will differ from the values assigned by equations (3.9). Since the values for $A$ and $B$ are bounded by 1, it follows that:

$$|\bar{A}_a(\lambda)| \leqslant 1 \qquad |\bar{B}_b(\lambda)| \leqslant 1. \qquad (3.10)$$

Using the general definition of locality of §3.3, we can write the expectation value for the product $A_a B_b$ as:

$$E(a, b) = \int_\Lambda \bar{A}_a(\lambda) \bar{B}_b(\lambda) \, \mathrm{d}\rho. \qquad (3.11)$$

Since we are including in our ensemble only those particles which have previously triggered the 'event-ready' detectors, we are assured that the distribution $\rho$ and the range of integration $\Lambda$ are independent of $a$ and $b$. Now consider the expression:

$$E(a, b) - E(a, b') = \int_\Lambda [\bar{A}_a(\lambda)\bar{B}_b(\lambda) - \bar{A}_a(\lambda)\bar{B}_{b'}(\lambda)] \, \mathrm{d}\rho$$

where we take $a'$ and $b'$ to be alternative settings for analysers 1 and 2, respectively. This can be rewritten as:

$$E(a, b) - E(a, b') = \int_\Lambda \bar{A}_a(\lambda)\bar{B}_b(\lambda)[1 \pm \bar{A}_{a'}(\lambda)\bar{B}_{b'}(\lambda)] \, \mathrm{d}\rho$$
$$- \int_\Lambda \bar{A}_a(\lambda)\bar{B}_{b'}(\lambda)[1 \pm \bar{A}_{a'}(\lambda)\bar{B}_b(\lambda)] \, \mathrm{d}\rho.$$

Using inequalities (3.10), we then have:

$$|E(a, b) - E(a, b')| \leqslant \int_\Lambda [1 \pm \bar{A}_{a'}(\lambda)\bar{B}_{b'}(\lambda)] \, \mathrm{d}\rho + \int_\Lambda [1 \pm \bar{A}_{a'}(\lambda)\bar{B}_b(\lambda)] \, \mathrm{d}\rho$$

or

$$|E(a, b) - E(a, b')| \leqslant \pm [E(a', b') + E(a', b)] + 2 \int_\Lambda \mathrm{d}\rho.$$

Hence:

$$-2 \leqslant E(a, b) - E(a, b') + E(a', b) + E(a', b') \leqslant 2. \qquad (3.12)$$

By re-definition of the parameters $a$, $a'$, $b$ and $b'$ in the central expression of (3.12), the minus sign may be permuted to any one of the four terms. Inequality (3.12) and its permutations are one form of Bell's inequality, and represent a general prediction for the theories covered by the above assumptions.

In order to complete the proof of the theorem, it is sufficient to show that in at least one situation the predictions by quantum mechanics contradict inequality (3.12). The quantum-mechanical prediction $[E(\hat{a}, \hat{b})]_{\mathrm{QM}}$ for the two spin-$\frac{1}{2}$ particle example, when due account is taken of imperfections in the analysers, detectors and state preparation, will be of the form:

$$[E(\hat{a}, \hat{b})]_{\mathrm{QM}} = C\hat{a} \cdot \hat{b} \qquad (3.13)$$

where the coefficient $C$ is bounded by one for actual systems, and is equal to plus or minus one only in the idealised case. Suppose we take $\hat{a}$, $\hat{a}'$, $\hat{b}$ and $\hat{b}'$ to be coplanar vectors as shown in figure 3 with $\phi = \pi/4$, and calculate:

$$[E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{b}') + E(\hat{a}', \hat{b}) + E(\hat{a}', \hat{b}')]_{QM} = 2\sqrt{2}C.$$

There is a wide range of values for $C$ for which the prediction by inequality (3.12) disagrees with that by equation (3.13). Hence the proof is complete.

### 3.5. The proof by Clauser and Horne

Clauser and Horne (1974) also proved Bell's theorem for general local realistic theories, including inherently stochastic theories. Their proof is noteworthy in that it
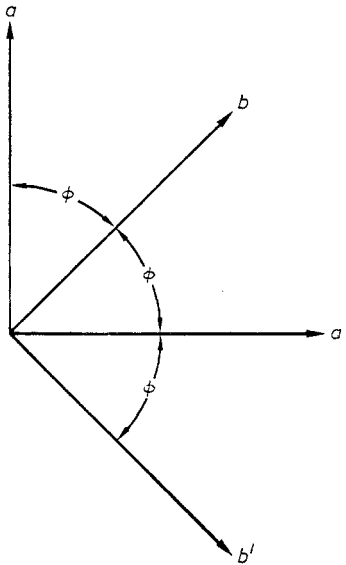


**Figure 3.** Optimal orientations for $a$, $a'$, $b$ and $b'$. If the correlation is of the form $C_1 + C_2 \cos n\phi$, then the maximum violation of the inequalities occurs at $n\phi = \pi/4$.

defines an experiment which might actually be performed and which does not require that auxiliary assumptions be made. The apparatus configuration which they used for the proof was introduced by Clauser *et al* (1969) and is shown schematically in figure 2. In contrast to the configuration of figure 1, theirs has only one detector in each arm and no 'event-ready' detectors. For each analyser/detector assembly there are only two possible outcomes: 'count' and 'no-count'. The results are thus formulated in terms of probabilities for single and coincidence counts, rather than the expectation values considered in §§3.1 and 3.4.

Suppose that during a period of time, while the adjustable parameters have the values $a$ and $b$, the source emits, say, $N$ of the two-component systems of interest. For this period, denote by $N_1(a)$ and $N_2(b)$ the number of counts at detectors 1 and 2, respectively, and by $N_{12}(a, b)$ the number of simultaneous counts from the two detectors (coincidence counts). When $N$ is sufficiently large, the probabilities for these

results for the whole ensemble (with due allowance for random errors) are given by:

$$p_1(a) = N_1(a)/N$$
$$p_2(b) = N_2(b)/N \qquad (3.14)$$
$$p_{12}(a, b) = N_{12}(a, b)/N.$$

CH derive an inequality which constrains ratios of the probabilities in equations (3.14) rather than their absolute magnitudes. Thereby, the influence of the quantity $N$ vanishes, so that it does not have to be measured.

Their derivation is straightforward. Following the discussion of §3.3, we expect a well-defined probability $p_1(\lambda, a)$ of detecting component 1, given the state $\lambda$ of the composite system and the parameter $a$ of the first analyser; a probability $p_2(\lambda, b)$ of detecting component 2, given $\lambda$ and $b$; and a probability $p_{12}(\lambda, a, b)$ of detecting both components, given $\lambda$, $a$ and $b$. Following our discussion of §3.3, we assume that, given $\lambda$, $a$ and $b$, the probabilities $p_1(\lambda, a)$ and $p_2(\lambda, b)$ are independent. Thus we write the probability of detecting both components as

$$p_{12}(\lambda, a, b) = p_1(\lambda, a)p_2(\lambda, b). \qquad (3.15)$$

The ensemble average probabilities of equations (3.14) are then given by:

$$p_1(a) = \int_\Lambda p_1(\lambda, a) \, d\rho$$
$$p_1(b) = \int_\Lambda p_2(\lambda, b) \, d\rho \qquad (3.16)$$
$$p_{12}(a, b) = \int_\Lambda p_1(\lambda, a)p_2(\lambda, b) \, d\rho.$$

To proceed, CH introduce the following lemma, the proof of which may be found in their paper: if $x, x', y, y', X, Y$ are real numbers such that $0 \leqslant x, x' \leqslant X$ and $0 \leqslant y, y' \leqslant Y$, then the inequality:

$$-XY \leqslant xy - xy' + x'y + x'y' - Yx' - Xy \leqslant 0 \qquad (3.17)$$

holds. Inequality (3.17) and equation (3.15) yield:

$$-1 \leqslant p_{12}(\lambda, a, b) - p_{12}(\lambda, a, b') + p_{12}(\lambda, a', b) + p_{12}(\lambda, a', b') - p_1(\lambda, a') - p_2(\lambda, b) \leqslant 0.$$
$$(3.18)$$

Integrating inequality (3.18) over $\lambda$ with distribution $\rho$, and using equation (3.16), one obtains the result:

$$-1 \leqslant p_{12}(a, b) - p_{12}(a, b') + p_{12}(a', b) + p_{12}(a', b') - p_1(a') - p_2(b) \leqslant 0. \qquad (3.19)$$

(Obtaining the left-hand inequality also required the use of equation (3.3), but the right-hand one did not. Since the left-hand inequality requires a measurement of the absolute magnitude of probabilities, the 'event-ready' detectors of figure 1 will be needed to test it.) The right-hand side of inequality (3.19) can be rewritten in the following form:

$$\frac{p_{12}(a, b) - p_{12}(a, b') + p_{12}(a', b) + p_{12}(a', b')}{p_1(a') + p_2(b)} \leqslant 1. \qquad (3.20(a))$$

As desired, it involves only a quantity that is independent of $N$. Using equations (3.14), and defining $R(a, b)$ as the rate of coincident detections, and $r_1(a)$ and $r_2(b)$ as

the rate of single-particle detections by either apparatus, inequality $(3.20(a))$ can be rewritten directly in terms of a ratio of observable count rates:

$$\frac{R(a,\, b) - R(a,\, b') + R(a',\, b) + R(a',\, b')}{r_1(a') + r_2(b)} \leqslant 1. \qquad (3.20(b))$$

Inequalities (3.20) are thus a general prediction for any local realistic theory of natural phenomena.

In order to complete the proof of the theorem it suffices to exhibit an instance in which the quantum-mechanical counterpart to inequalities (3.20) fails. This is done in §4, when we discuss the experimental requirements for a valid test of these theories.

### 3.6. Symmetry considerations

Almost all of the experiments which have been proposed for testing the predictions by Bell's inequalities involve pairs of polarised particles (either photons or massive particles). In these experiments the parameters $a$ and $b$, considered abstractly in §§3.4 and 3.5, are taken to be orientation angles relative to some reference axis in a fixed plane. In most of these experiments, the method of preparing the pairs of polarised particles attempts to achieve cylindrical symmetry about a normal to the fixed plane and reflection symmetry with respect to planes through this normal. This symmetry is exhibited in the quantum-mechanical predictions for detection rates and correlations:

$[p_1(a)]_{QM}$ and $[r_1(a)]_{QM}$ are independent of $a$.

$[p_2(b)]_{QM}$ and $[r_2(b)]_{QM}$ are independent of $b$.

$[p_{12}(b)]_{QM}$, $[R(a, b)]_{QM}$ and $[E(a, b)]_{QM}$ are functions only of $|a-b|$.

We now assume that the corresponding predictions for local realistic theories exhibit the same symmetries:

$$p_1(a) \equiv p_1 \text{ and } r_1(a) \equiv r_1 \text{ are independent of } b$$

$$p_2(b) \equiv p_2 \text{ and } r_2(b) \equiv r_2 \text{ are independent of } b \qquad (3.21)$$

$$p_{12}(a, b) \equiv p_{12}(|a-b|), \ R(a, b) \equiv R(|a-b|) \text{ and } E(a, b) \equiv E(|a-b|).$$

We must emphasise two points concerning equation (3.21). First, these symmetry relations do not simply follow from the corresponding quantum-mechanical symmetry relations or from the symmetry of the experimental arrangement, for one does not know what symmetry-breaking factors may lurk at the level of the hidden variables. Second, no harm is done in assuming equations (3.21), since they are susceptible to experimental verification.

Now suppose that we take $a$, $a'$, $b$ and $b'$ so that:

$$|a-b| = |a'-b| = |a'-b'| = \tfrac{1}{3}|a-b'| = \phi$$

as in figure 3. With the use of equation (3.21), inequalities (3.12) and (3.20) simplify to

$$|3E(\phi) - E(3\phi)| \leqslant 2 \qquad (3.22)$$

and

$$S(\phi) \leqslant 1 \qquad (3.23)$$

where we have defined:

$$S(\phi) \equiv \frac{3p_{12}(\phi) - p_{12}(3\phi)}{p_1 + p_2} \qquad (3.24(a))$$

in terms of probabilities, or equivalently in terms of count rates:

$$S(\phi) \equiv \frac{3R(\phi) - R(3\phi)}{r_1 + r_2}. \qquad (3.24(b))$$

### 3.7. *The proof by Wigner, Belinfante and Holt*

A simple method of proving Bell's theorem for deterministic local hidden-variables theories was invented independently by Wigner (1970) and Belinfante (1973), and extended by Holt (1973). The method consists of subdividing the space $\Lambda$ of states of a two-component system into subspaces corresponding to various possible values of the observables of interest, and then performing some easy calculations on the measures of these subspaces. Rather than duplicate their proofs, which are readily available, we show how their method can be used to derive the inequality of CH.

Consider the apparatus configuration of figure 2. Assume that the detection or non-detection of component 1 is completely determined by the parameter $a$ of the first analyser and the state of the composite system, but is independent of the parameter $b$ of the other analyser, and so forth for component 2. As such, the discussion is for the restricted situation in which determinism applies. Under this assumption, we can exhaustively subdivide the space $\Lambda$ into 16 mutually disjoint subspaces $\Lambda$ $(ij; kl)$, where each letter can take on the value 0 or 1, with 1 denoting detection and 0 non-detection; with $i$ and $j$ referring to the results if the parameter of the first analyser is chosen respectively to be $a$ or $a'$; and with $k$ and $l$ referring to the results if the parameter of the second analyser is chosen respectively to be $b$ or $b'$. For example, $\Lambda$ (10; 01) is the subspace in which component 1 will be detected if its associated parameter is chosen to be $a$ but will not if the parameter is chosen to be $a'$, while component 2 will not be detected if its associated parameter is chosen to be $b$ but will be detected if that parameter is chosen to be $b'$. (Note that there is no question of simultaneously examining detection or non-detection for two different values of a parameter. Indeed, such observations are mutually exclusive. Rather, the subspace is defined in terms of what will happen if any one of the various experiments is performed. Since the theories are assumed to be deterministic, these values are all determined once $a$, $b$, $\lambda$ and the apparatus configuration are specified.) If a probability measure $\rho$ is assumed to be given on $\Lambda$ (determined presumably by the way in which the composite system is prepared), then $\rho(ij; kl)$ is defined to be the probability that the composite state is in $\Lambda(ij; kl)$. Clearly, all $\rho(ij; kl)$ are non-negative. Because the 16 subspaces are disjoint and exhaustive, we have:

$$\sum_{ijkl} \rho(ij; kl) = 1. \qquad (3.25)$$

We now define $p_1(a)$ to be the probability that component 1 will be detected if its parameter is chosen to be $a$; $p_2(b)$ to be the probability that component 2 will be detected if its parameter is chosen to be $b$; and $p_{12}(a, b)$ to be the probability of joint detection of both components if the two parameters are chosen respectively to be $a$ and $b$. Analogous definitions are given for the other values of the parameters. Then

we have:

$$p_{12}(a, b) = \rho(11; 11) + \rho(11; 10) + \rho(10; 11) + \rho(10; 10)$$

$$p_{12}(a, b') = \rho(11; 11) + \rho(11; 01) + \rho(10; 11) + \rho(10; 01)$$

$$p_{12}(a', b) = \rho(11; 11) + \rho(11; 10) + \rho(01; 11) + \rho(01; 10)$$

$$p_{12}(a', b') = \rho(11; 11) + \rho(11; 01) + \rho(01; 11) + \rho(01; 01) \quad (3.26)$$

$$p_1(a') = \rho(11; 11) + \rho(11; 10) + \rho(11; 01) + \rho(11; 00)$$
$$+ \rho(01; 11) + \rho(01; 10) + \rho(01; 01) + \rho(01; 00)$$

$$p_2(b) = \rho(11; 11) + \rho(11; 10) + \rho(10; 11) + \rho(10; 10)$$
$$+ \rho(01; 11) + \rho(01; 10) + \rho(00; 11) + \rho(00; 10).$$

It follows that:

$$p_{12}(a, b) - p_{12}(a, b') + p_{12}(a', b) + p_{12}(a', b') - p_1(a') - p_2(b)$$
$$= - \rho(11; 01) - \rho(11; 00) - \rho(10; 11) - \rho(10; 01) - \rho(01; 10)$$
$$- \rho(01; 00) - \rho(00; 11) - \rho(00; 10). \quad (3.27)$$

Consequently, we recover inequality (3.19) derived by Clauser and Horne for the more general stochastic case:

$$-1 \leqslant p_{12}(a, b) - p_{12}(a, b') + p_{12}(a', b) + p_{12}(a', b') - p_1(a') - p_2(b) \leqslant 0.$$

The demonstration of the incompatibility between this inequality and quantum mechanics is thus the same as that of § 3.5, and hence the theorem is proved.

### 3.8. Stapp's proof

Stapp's version of Bell's theorem (1971, 1977) appears to be very general, for it dispenses with all assumptions about the state of the system and about probability measures on the space of states. The proof was generalised by Eberhard (1977) to include realisable systems. Stapp considered a long series of $N$ occurrences of Bohm's *Gedankenexperiment*. In each occurrence a pair of spin-$\frac{1}{2}$ particles is produced in the singlet state in a space–time region $S_0$. The particles propagate in opposite directions along a given axis. Particle 1 proceeds to a space–time region $S_1$, where it is deflected 'up' or 'down' by a Stern–Gerlach magnet oriented in either the $\hat{a}$ or the $\hat{a}'$ direction, and particle 2 proceeds to the region $S_2$ where it is deflected up or down by a magnet oriented in either the $\hat{b}$ or the $\hat{b}'$ direction. $S_1$ and $S_2$ are supposed to have space-like separation, and the choice for orienting the first magnet along $\hat{a}$ or $\hat{a}'$ is made when particle 1 is in $S_1$, and similarly with the choice for orienting the second magnet. Let the number 1 or $-1$ be recorded for a particle entering the field of a Stern–Gerlach magnet accordingly as it is deflected 'up' or 'down'. Let $r_{\alpha j}(\hat{a}, \hat{b})$ (where $\alpha = 1, 2$ and $j = 1, \ldots, N$) be the number recorded for the $\alpha$th particle of the $j$th pair if the two magnets are oriented in the $\hat{a}$ and $\hat{b}$ directions respectively, and let $r_{\alpha j}(\hat{a}, \hat{b}')$, $r_{\alpha j}(\hat{a}', \hat{b})$ and $r_{\alpha j}(\hat{a}', \hat{b}')$ have analogous meanings. Clearly the orientations $\hat{a}$ and $\hat{a}'$ are mutually exclusive, as are $\hat{b}$ and $\hat{b}'$. Although only one of the four possible pairs of orientations $(\hat{a}, \hat{b})$, $(\hat{a}, \hat{b}')$, $(\hat{a}', \hat{b})$, $(\hat{a}', \hat{b}')$ can occur in the real world, Stapp made an assumption

of 'counterfactual definiteness', that $r_{\alpha j}(\hat{a}, \hat{b})$, $r_{\alpha j}(\hat{a}, \hat{b}')$, etc, are all definite numbers. In addition, he made an assumption of individual locality, that:

$$r_{1j}(\hat{a}, \hat{b}) = r_{1j}(\hat{a}, \hat{b}') \qquad (3.28(a))$$

$$r_{1j}(\hat{a}', \hat{b}) = r_{1j}(\hat{a}', \hat{b}') \qquad (3.28(b))$$

$$r_{2j}(\hat{a}, \hat{b}) = r_{2j}(\hat{a}', \hat{b}) \qquad (3.28(c))$$

$$r_{2j}(\hat{a}, \hat{b}') = r_{2j}(\hat{a}', \hat{b}'). \qquad (3.28(d))$$

Stapp then showed that the $8N$ numbers $r_{\alpha j}(\hat{a}, \hat{b})$, etc, must disagree with some of the statistical predictions of quantum mechanics. Some critics of Stapp have argued that his assumption of counterfactual definiteness is understandable only from the standpoint of a deterministic local hidden-variables theory. Stapp (1978, §4) has replied, however, that his assumption requires no commitment to determinism, but only to the possibility of speaking (as is commonly done in the sciences) of possible worlds as well as the actual one. He makes the explicit assumption that each of the four choices $(\hat{a}, \hat{b})$, $(\hat{a}, \hat{b}')$, $(\hat{a}', \hat{b})$ and $(\hat{a}', \hat{b}')$ is made in some possible world. It may nevertheless be objected that Stapp has not given a reason for demanding the existence of a quadruple of possible worlds which mesh together as in equations $(3.28(a)$–$(d))$. The combination of no action-at-a-distance with the idea of possible worlds only seems to require four pairs of possible worlds, one pair meshing as in equation $(3.28(a))$, one as in equation $(3.28(b))$, etc. It is not obvious why these four relations need to govern a cluster of four possible worlds unless determinism is supposed. An answer to this objection is provided by Stapp (1978), in which the following equivalence theorem is proved.

Let $I$ be the set of individual outcomes $r_{\alpha j}(c, d)$, where $c$ is $\hat{a}$ or $\hat{a}'$, $d$ is $\hat{b}$ or $\hat{b}'$, $\alpha$ is 1 or 2, and $j$ is 1, . . . , $N$. Let $P(I)$ be the set of probabilities

$$P = (\{r_1 \mid \hat{a}\}, \{r_2 \mid \hat{b}\}, \{r_1, r_2 \mid \hat{a}, \hat{b}\})$$

determined by the appropriate frequencies in $I$:

$$\{r_1 \mid \hat{a}\} = N(r_1, \hat{a})/N$$

(where $N(r_1, \hat{a})$ is the number of $j$ such that $r_1 = r_{1j}(\hat{a}, \hat{b}) = r_{1j}(\hat{a}, \hat{b}')$ by individual locality),

$$\{r_1, r_2 \mid \hat{a}, \hat{b}\} = N(r_1, r_2, \hat{a}, \hat{b})$$

(where $N(r_1, r_2, \hat{a}, \hat{b})$ is the number of $j$ such that $r_1 = r_{1j}(\hat{a}, \hat{b})$ and $r_2 = r_{2j}(\hat{a}, \hat{b})$), etc.

Let $L_P$ be the set of $P$ which satisfy the following probabilistic locality conditions: there exists a discrete set of $\lambda$, a probability weight function $\rho$ defined on this set, and probabilities $p_1(\lambda, \hat{a}, r_1)$, $p_2(\lambda, \hat{b}, r_2)$ for the outcomes $r_1$ and $r_2$ respectively (given $\lambda$ and given $\hat{a}$ or $\hat{b}$), such that:

$$\{r_1, r_2 \mid \hat{a}, \hat{b}\} = \sum_\lambda \rho(\lambda) p_1(\lambda, \hat{a}, r_1) p_2(\lambda, \hat{b}, r_2)$$

$$\{r_1 \mid \hat{a}\} = \sum_\lambda \rho(\lambda) p_1(\lambda, \hat{a}, r_1) \qquad (3.29)$$

$$\{r_2 \mid \hat{b}\} = \sum \rho(\lambda) p_1(\lambda, \hat{b}, r_2).$$

Finally, let $L$ be the set of $I$ which satisfy the individual locality conditions $(3.28(a)$–$(d))$. Then the equivalence theorem asserts (i) if $I \in L$, then $P(I) \in L_P$, (ii) if $P \in L_P$, then there is an $I \in L$ such that $P(I)$ is approximately equal to $P$.

Note that (3.29) is essentially the CH probabilistic locality condition, except that a sum over discrete values of $\lambda$ is used instead of an integral over the space $\Lambda$; but since the integral can always be approximated by a sum, this difference is not crucial. Because of this equivalence theorem, the theorem of CH and of Bell that no $P$ which belongs to $L_P$ can agree statistically with quantum mechanics entails that no $I$ which belongs to $L$ can agree statistically with quantum mechanics and, conversely, the theorem of Stapp that no $I$ belonging to $L$ can agree statistically with quantum mechanics implies the theorem of CH and Bell. Stapp's equivalence theorem, therefore, shows that, contrary to appearances, his proof of Bell's theorem and those of CH and Bell (1971) are of equal strength. J S Bell (personal communication) has remarked that part (ii) of Stapp's equivalence theorem is an example of the possibility of simulating a stochastic process with a deterministic one.

### 3.9. Other versions of Bell's theorem

Several other versions of Bell's theorem have been discovered. The proofs are mathematically correct, but with hypotheses in some respects problematic, either from a philosophical point of view or from their inherent restriction to idealised systems.

A very general derivation of Bell's theorem has been presented by Bell (1976). It was critically evaluated by Shimony *et al* (1976), who challenged one of the premises. Bell (1977) replied to this criticism. If we retain the notation of §3.8, we can express the essential assumption of Bell (1976) in the following way: the complete state of region $S_1$ is independent of the choice between $\hat{b}$ and $\hat{b}'$ in $S_2$, and likewise the complete state of $S_2$ is independent of the choice between $\hat{a}$ and $\hat{a}'$ in $S_1$. Shimony *et al* (1976) criticised this assumption on the ground that the backward light cones of $S_1$ and $S_2$ overlap in a region $S$, and it is possible that a factor in $S$ affecting the choice between $\hat{b}$ and $\hat{b}'$ leaves some trace in $S_1$. Bell's reply (1977) to this objection stresses the spontaneity of the experimenter's choice between $\hat{b}$ and $\hat{b}'$ and between $\hat{a}$ and $\hat{a}'$, but this answer seems to us to depend upon too strong a commitment to indeterminism for his argument to be fully general (see also Shimony 1978).

The proof by d'Espagnat (1975) has the virtue of staying quite close to the original ideas of EPR by reasoning in terms of the intrinsic properties of the system. We shall not try to summarise his argument, partly because of its length and partly because of a premise which is impossible to be realised experimentally. D'Espagnat assumes (as Bell did in 1965) that one has a system like a pair of spin-$\frac{1}{2}$ particles in the singlet state, such that one can measure an observable of one of the pair and then infer with absolute certainty the value of a corresponding observable of the other pair (equation (3.2)). The same criticism can also be made of the arguments of Gutkowski and Masotto (1974), Selleri (1978) and Schiavulli (1977) but it should be noted that they derive a number of generalisations of Bell's inequalities which have not been obtained elsewhere.

## 4. Considerations regarding a general experimental test

Following Bell's (1965) results, many readers believed that local realistic theories were *ipso facto* discredited, because quantum mechanics has been so abundantly confirmed in a variety of experimental situations. Indeed, some of the most striking

confirmations of quantum mechanics, such as the spectrum of helium, concerned correlated pairs of particles. However, upon careful examination, one finds that situations exhibiting the disagreement discovered by Bell are rather rare, and none had ever been experimentally realised. Moreover, the reasoning of the previous sections indicates that the treatment of correlated but spatially separated systems may well be the point of greatest vulnerability of quantum mechanics. In view of the consequences of Bell's theorem it is thus important to design experiments to test explicitly the predictions made for local realistic theories via Bell's theorem.

Starting with the simple configurations specified in §3, the first problem is to find a suitable system whose quantum-mechanical predictions directly violate the predictions in the theorem, but additionally one that is accessible with available technology. In fact, this has not yet been done! (Possible avenues in this direction are discussed in §7.) In the present section we examine the requirements for a fully general test, and see why the problem is difficult. Since the presence of auxiliary counters is required by the apparatus configuration of figure 1, and usually these depolarise or destroy the emissions, we will confine our discussion to the apparatus configuration of figure 2.

We thus compare the quantum-mechanical predictions for this configuration with those by inequality (3.23). The left-hand side of inequality (3.19) is not considered here, since it cannot be expressed in terms of ratios of observable probabilities. It will, however, become useful for the discussion of §5.

### 4.1. Requirements for a general experimental test

Consider an experiment, with a configuration similar to that of figure 2, whose quantum-mechanical predictions take the following form:

$$[p_{12}(\phi)]_{\mathrm{QM}} = \tfrac{1}{4}\,\eta_1\eta_2 f_1 g[\epsilon_+^1\epsilon_+^2 + \epsilon_-^1\epsilon_-^2\, F\cos(n\phi)]$$
$$[p_1]_{\mathrm{QM}} = \tfrac{1}{2}\,\eta_1 f_1\epsilon_+^1 \qquad\qquad (4.1)$$
$$[p_2]_{\mathrm{QM}} = \tfrac{1}{2}\,\eta_2 f_2\epsilon_+^2.$$

This general form is characteristic of the quantum-mechanical predictions for the actual experiments of interest (see, for example, equation (5.15)). In these expressions $\eta_i$ represents the effective quantum efficiency of detector $i(i=1, 2)$, and

$$\epsilon_+^i \equiv \epsilon_{\mathrm{M}}^i + \epsilon_{\mathrm{m}}^i \qquad\qquad \epsilon_-^i \equiv \epsilon_{\mathrm{M}}^i - \epsilon_{\mathrm{m}}^i. \qquad\qquad (4.2)$$

The terms $\epsilon_{\mathrm{M}}^i$ and $\epsilon_{\mathrm{m}}^i$ are the maximum and minimum transmissions of the analysers relative to the pertinent orthogonal basis. The functions $f_1$ and $f_2$ are the collimator efficiencies, i.e. the probability that an appropriate emission enters apparatus 1 or 2. Typically, these are simply proportional to the collimator acceptance solid angles. The function $g$ is the conditional probability that, given emission 1 enters apparatus 1, then emission 2 will enter apparatus 2. The function $F$ is a measure of the initial-state purity and the inherent quantum-mechanical correlation of the two emissions. For the actual cascade-photon experiments (see §5), these functions depend on the collimator solid angles. The values of $n$ are 1 or 2 depending upon whether the experiment is performed with fermions or bosons.

Inserting equation (4.1) into the definition of $S(\phi)$, equation (3.24(a)), we find the quantum-mechanical prediction for this function to be given by:

$$S_{\mathrm{QM}}(\phi) = \tfrac{1}{4}\,\eta g\,\{2\epsilon_+ + [3\cos(n\phi) - \cos(3n\phi)]\,F(\epsilon_-^2/\epsilon_+)\}. \qquad\qquad (4.3)$$

Here for simplicity we have taken $\eta \equiv \eta_1 = \eta_2$, $f_1 = f_2$, $\epsilon_+ = \epsilon_+^1 = \epsilon_+^2$ and $\epsilon_- \equiv \epsilon_-^1 = \epsilon_-^2$. Selecting the optimum value $\phi = \pi/4n$, one finds that the condition for a violation of inequality (3.23) is given by:

$$\eta g \epsilon_+ [\sqrt{2}\,(\epsilon_-/\epsilon_+)^2\, F + 1] > 2. \qquad (4.4)$$

Thus, a correlation experiment with values in the domain specified by inequality (4.4) is capable of distinguishing between the prediction, inequality (3.23), and that of quantum theory, equation (4.3). Although such experiments are apparently possible, there is at present no existing experimental result in this domain, and thus none in violation of any inequality which does not require additional assumptions for its derivation.

For a direct test of inequality (4.4) the requirements are stringent, which accounts for the fact that, so far, no such experiment has been attempted.

(i) A source must emit pairs of discrete-state systems, which can be detected with high efficiency.

(ii) Quantum mechanics must predict strong correlations of the relevant observables of each pair (polarisations in the experiments so far). Correspondingly, the ensemble of pairs must have high quantum-mechanical purity.

(iii) The analysers must be capable of allowing systems in certain states to pass with great efficiency, while simultaneously rejecting nearly all of those in orthogonal states.

(iv) The collimators (and filters if these are necessary to remove unwanted emissions, etc) must have very high transmittances and not depolarise the emissions.

(v) The source must produce the systems via a two-body decay. A three- (or more) body decay cannot be used, because the resulting angular correlation will make $g \ll 1$.

(vi) Another requirement should be added in order to achieve an airtight argument against locality: the parameters $a$ and $b$ must be rapidly changed while the emissions are in flight. A detection event should be space-like separated from the corresponding parameter change event at the far apparatus (see §7). This suggestion was first made by Bohm and Aharonov (1957).

For a practical experiment, it is of course also necessary for the counting rate to be sufficiently high to make the required integration time reasonable.

### 4.2. Three important experimental cases

Let us examine how the failure of any of these parameters to approximate the ideal case prevents a violation of inequality (3.23) from arising. Figure 4 shows the prediction by equation (4.3) for three important cases of interest, along with the prediction by inequality (3.23).

*Case I, nearly ideal.* In the domain of nearly ideal apparatus, we have $g \approx \epsilon_+ \approx \epsilon_- \approx \eta \approx 1$. For these conditions we find a violation of inequality (3.23) for a wide range of $\phi$, with a maximum violation at $n\phi = \pi/4$.

*Case II, poor detector efficiencies or co-focusing.* When $g \ll 1$ holds, because of imperfect collimator alignment and/or a weak angular correlation inherent in a three-body decay, or when $\eta \ll 1$ holds, because the detector efficiencies are low, then the amplitude of $S(\phi)$ contracts in amplitude about a value close to zero. The quantum-mechanical predictions enter a domain where no violation of inequality (3.23) occurs.
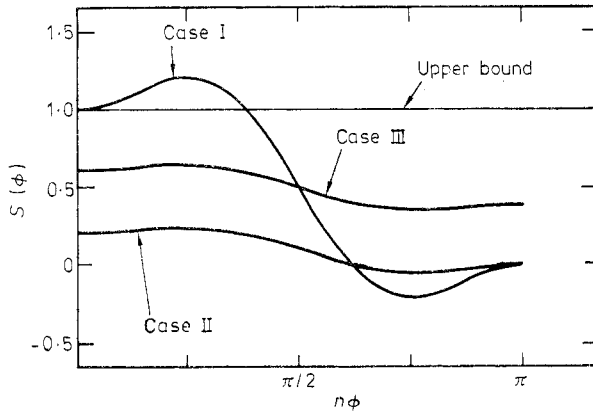
**Figure 4.** Typical dependence of $S(\phi)$ upon $n\phi$, for cases I–III. Upper bound for $S(\phi)$ set by inequality (3.23) is $+1$. Case I experiments (nearly ideal) have $\eta \approx g \approx F \approx \epsilon_+ \approx \epsilon_- \approx 1$. Case II experiments have nearly ideal parameters $F \approx \epsilon_- \approx \epsilon_+ \approx 1$, but have $\eta \ll 1$ and/or $g \ll 1$. Case III experiments have nearly ideal parameters $\eta \approx g \approx 1$, but have $F \ll 1$ and/or $\epsilon_-/\epsilon_+ \ll 1$.

This case is typical of the low-energy cascade-photon experiments to be described in §5.

*Case III, weak correlation.* The third case occurs when the predicted correlation is weak. The correlation coefficient $F$ and/or the parameter $\epsilon_-/\epsilon_+$ may be much less than unity. This will occur, for example, if the emissions are only weakly correlated, if the initial state is impure, if the emissions suffer significant depolarisation in passing through the apparatus, or if the analyser efficiencies are low. The curve $S(\phi)$ then contracts in amplitude symmetrically about a value slightly less than $+\frac{1}{2}$, and again no violation of inequality (3.23) occurs. This case is typical of the positronium annihilation and the proton–proton S-wave scattering experiments to be described in §6.

The manner in which the amplitude of $S(\phi)$ contracts is of more importance than it may seem. To perform a test of the local realistic theories in the domain of case II and III experiments requires a credible auxiliary assumption that $S(\phi)$ can be rescaled somehow to an amplitude sufficient to violate the inequalities. Case II experiments (discussed in §5) are more favourable in this respect than are those of case III. For the former, a replacement of $p_1$ and $p_2$ (singles rates) with carefully selected coincidence rates can provide this rescaling at the small price of accepting only a very mild auxiliary assumption. On the other hand, rescaling case III experiments (see §6) requires one to assume a certain *ad hoc* modification of the basic correlation coefficient $F$. However, the measurement of this coefficient is in many respects a primary objective of the experiment. Any such assumption must then be scrutinised very carefully, for it inherently becomes the weak point of the experiment.

## 5. Cascade-photon experiments

The essential problem in testing the predictions in Bell's theorem against those by quantum mechanics is to find experimentally realisable situations in which the

quantum-mechanical predictions directly violate Bell's inequalities. In §4 we showed that to do so with available apparatus is difficult. The situation is not hopeless, however. Clauser *et al* (1969) showed that with a mild supplementary assumption, actual experiments are predicted by quantum mechanics to yield a violation of Bell's inequality, and they proposed such an experiment.

Their suggestion is to measure the correlation in linear polarisation of photon pairs emitted in an atomic cascade. Figure 5 shows a schematic diagram of a typical apparatus for doing this, that of Freedman and Clauser (1972), who reported the first such test. The photons were emitted in a $J = 0 \rightarrow J = 1 \rightarrow J = 0$ atomic cascade. The decaying atoms were viewed by two symmetrically placed optical systems, each consisting of two lenses, a wavelength filter, a rotatable and removable polariser, and a single-photon detector. The following quantities were measured: $R(\phi)$, the coincidence rate for two-photon detection as a function of the angle $\phi$ between the planes of linear polarisation, defined by the orientations of the inserted polarisers; $R_1$, the
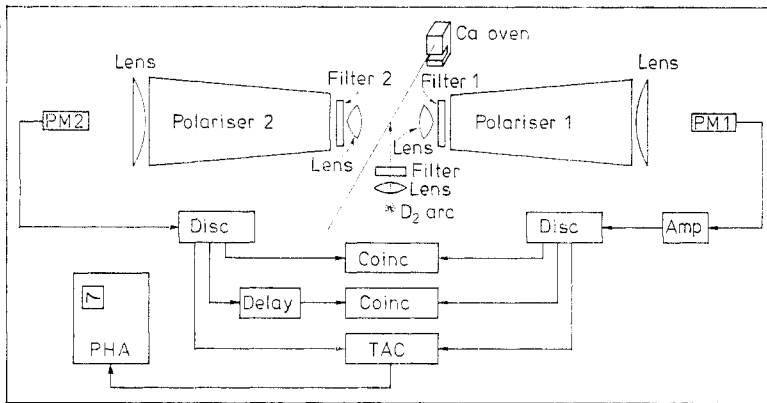


**Figure 5.** Schematic diagram of apparatus and associated electronics of the experiment by Freedman and Clauser. Scalers (not shown) monitored the outputs of the discriminators and coincidence circuits (figure after Freedman and Clauser).

coincidence rate with polariser 2 removed; $R_2$, the coincidence rate with polariser 1 removed; $R_0$, the coincidence rate with both polarisers removed.

The details of this experiment along with other similar ones will be discussed in §5.3. First, however, we describe the auxiliary assumption(s) which render this a reasonable test, and present the resulting inequalities. Then we describe the quantum-mechanical predictions for this and similar arrangements.

### 5.1. Predictions by local realistic theories

*5.1.1. Assumptions for cascade-photon experiments.* The initial assumption by CHSH is, given that a pair of photons *emerges* from the polarisers, the probability of their joint detection is independent of the polariser orientations $a$ and $b$. Clauser and Horne (1974) showed that an alternative assumption leads to the same results. Their assumption is that for every pair of emissions (i.e. for each value of $\lambda$), the probability of a count with a polariser in place is less than or equal to the corresponding probability with the polariser removed. The assumption of CH is stronger than that of CHSH in so far as it is stated for each value of $\lambda$, whereas CHSH make an assertion only for

the total sub-ensemble of photons which pass through the polarisers. On the other hand, the assumption of CH is more general, in that the processes 'passage' and 'non-passage' through a polariser (which are not observable, and which are inappropriate for many possible theories) are not considered primitive. Furthermore, CH only assume an inequality, which is weaker than the equality of CHSH. Both assumptions, in our opinion, are physically plausible, but each gives a certain loophole to those who wish to defend local hidden-variables theories in spite of the experimental evidence which will be presented below.

Let us discuss the consequences of the CH assumption. We denote by the symbol $\infty$ an apparatus configuration in which the analyser is absent. Let $p_1(\lambda, \infty)$ denote the probability of a count from detector 1 when analyser 1 is absent and the state of the emission is $\lambda$. A similar probability $p_2(\lambda, \infty)$ may be defined for apparatus 2. Thus, the assumption is that:

$$0 \leqslant p_1(\lambda, a) \leqslant p_1(\lambda, \infty) \leqslant 1$$
$$0 \leqslant p_2(\lambda, b) \leqslant p_2(\lambda, \infty) \leqslant 1 \tag{5.1}$$

for every $\lambda$, and for all values of $a$ and $b$. Inequalities (5.1) and (3.17) and arguments similar to those which led from (3.16) to (3.19) yield immediately the result:

$$-p_{12}(\infty, \infty) \leqslant p_{12}(a, b) - p_{12}(a, b') + p_{12}(a', b) + p_{12}(a', b') - p_{12}(a', \infty) - p_{12}(\infty, b) \leqslant 0. \tag{5.2}$$

Note that all terms in inequality (5.2) are joint probabilities for coincident counts at the two detectors. Inequality (3.19), in contrast, contains the two terms $p_1$ and $p_2$, which are probabilities of a count at a single detector. Furthermore, both the upper and lower limits in inequality (5.2) can be written as a ratio of probabilities, so that both can be tested without the need for the 'event-ready' detectors of figure 1.

*5.1.2. Additional symmetries.* Again we can invoke a rotational invariance argument similar to that of §3.6; thus we require:

  (i) $p_{12}(a, \infty)$ is independent of $a$, and likewise $R_1(a) \equiv R_1$
  (ii) $p_{12}(\infty, b)$ is independent of $b$, and likewise $R_2(b) \equiv R_2$
  (iii) $p_{12}(a, b) \equiv p_{12}(\phi)$, and likewise $R(a, b) \equiv R(\phi)$, where $\phi = |a - b|$.

These conditions are not always satisfied, and they obviously fail when each of the particles has a definite linear polarisation. However, for all of the actual experiments to be described in this section, the conditions are at least satisfied by the quantum-mechanical predictions, and more importantly no experimental deviations from them have been detected. It is noteworthy that this set of conditions is frequently satisfied, even in situations where some of those of §3.6 are not. For example, in many of the cascade-photon experiments, the singles rate $r_2$ contains an extraneous contribution from excitation to the intermediate state of the cascade by channels not involving the first level of the cascade. Such excitation may result in the emission of polarised light at the wavelength of the second photon of the cascade, but no coincidences.

With these conditions, inequality (5.2) becomes:

$$-p_{12}(\infty, \infty) \leqslant 3p_{12}(\phi) - p_{12}(3\phi) - p_{12}(a', \infty) - p_{12}(\infty, b) \leqslant 0 \tag{5.3}$$

for all $a'$ and $b$.

Since the emission rates in all of the various experiments were held constant, and

in most cases monitored by an auxiliary apparatus, we can write the ratios of probabilities as ratios of count rates:

$$p_{12}(\phi)/p_{12}(\infty, \infty) = R(\phi)/R_0$$

$$p_{12}(a, \infty)/p_{12}(\infty, \infty) = R_1/R_0 \qquad (5.4)$$

$$p_{12}(\infty, b)/p_{12}(\infty, \infty) = R_2/R_0.$$

Inserting equations (5.4) into inequality (5.3), we can write this form of Bell's inequality in terms of coincidence rates:

$$-R_0 \leqslant 3R(\phi) - R(3\phi) - R_1 - R_2 \leqslant 0. \qquad (5.5)$$

Inequality (5.5) was first derived by Clauser *et al* (1969), but by using their alternative auxiliary assumption.

Freedman (1972) showed that inequality (5.5) can be further contracted to a form which is very convenient for comparison with experimental results. If we take the optimal value for upper-limit violation by cascade-photon experiments $\phi = \pi/8$, then inequality (5.5) becomes:

$$-R_0 \leqslant 3R(\pi/8) - R(3\pi/8) - R_1 - R_2 \leqslant 0.$$

On the other hand, if we take the optimal value for lower-limit violation $\phi = 3\pi/8$, using the fact that $9\pi/8$ represents the same angle as $\pi/8$, it becomes:

$$-R_0 \leqslant 3R(3\pi/8) - R(\pi/8) - R_1 - R_2 \leqslant 0.$$

Dividing both inequalities by $R_0$, and subtracting the second inequality from the preceding one, we obtain the simple inequality:

$$|R(\pi/8) - R(3\pi/8)|/R_0 \leqslant \tfrac{1}{4}. \qquad (5.6)$$

Inequality (5.6) has the advantage that it can be checked by measuring the frequency of joint detection of photons with the polarisers in only two different relative orientations, and it dispenses with the need to measure rates with only one polariser removed.

### 5.2. Quantum-mechanical predictions for a $J = 0 \to 1 \to 0$ two-photon correlation

*5.2.1. An idealised case.* Even if ideal polarisation analysers and photo-detectors are assumed, the violation or non-violation of inequality (5.6) depends upon the quantum state in which the photon pairs are prepared. It is instructive to demonstrate that a violation does occur with perfect apparatus if the photons are propagating in opposite directions from the source along the $\hat{z}$ axis, with total angular momentum 0 and total parity $+1$. Their state is an ideal limit of ones which can actually be prepared in a laboratory. The polarisation part of the two-photon wavefunction is:

$$\Psi_0 = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \qquad (5.7)$$

where $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ represents polarisation along the $\hat{x}$ axis and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ represents polarisation along the $\hat{y}$ axis, and where the first of two juxtaposed column vectors refers to photon 1 and the second to photon 2. A projection operator for linear polarisation along an

axis, lying in the $xy$ plane and making an angle $\theta$ with the $\hat{x}$ axis is:

$$Q(\theta) \equiv \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (5.8)$$

as one can check by noting that the vector $\left(\begin{smallmatrix}\cos\theta\\\sin\theta\\0\end{smallmatrix}\right)$, which represents linear polarisation in this direction, is an eigenvector of $Q(\theta)$ with eigenvalue 1. Similarly the vector $\left(\begin{smallmatrix}-\sin\theta\\\cos\theta\\0\end{smallmatrix}\right)$, representing linear polarisation perpendicular to this direction, is an eigenvector of $Q(\theta)$ with eigenvalue 0, as is $\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$ which represents polarisation along the $z$ axis (which of course is excluded by transversality). Consequently, the quantum-mechanical prediction for this case is:

$$[R(\phi)/R_0]_{\Psi_0} = \langle \Psi_0 | Q(a) \otimes Q(b) | \Psi_0 \rangle = \tfrac{1}{4}(1 + \cos 2\phi) \qquad (5.9)$$

where, as before, we have taken $\phi = |a - b|$. From this result we find that the quantum-mechanical predictions

$$[R(\pi/8)/R_0 - R(3\pi/8)/R_0]_{\Psi_0} = \tfrac{1}{4}\sqrt{2} \qquad (5.10)$$

violate inequality (5.5).

*5.2.2. Quantum-mechanical predictions for $J = 0 \rightarrow 1 \rightarrow 0$ cascade, ideal analysers, and finite solid-angle detectors.* Consider a $J = 0 \rightarrow J = 1 \rightarrow J = 0$ atomic cascade in which no angular momentum is exchanged with the nucleus, and in which both transitions are electric dipole. Since the atom is both initially and finally in states with zero total angular momentum, and since there is a parity change in each transition, the emitted photon pair has zero total angular momentum and even parity. We can therefore exactly write the angular wavefunction of the photon pair as:

$$\Psi = \frac{1}{\sqrt{3}} [Y_{1,1}{}^1(\hat{\eta}_1) Y_{1,-1}{}^1(\hat{\eta}_2) - Y_{1,0}{}^1(\hat{\eta}_1) Y_{1,0}{}^1(\hat{\eta}_2) + Y_{1,-1}{}^1(\hat{\eta}_1) Y_{1,1}{}^1(\hat{\eta}_2)] \qquad (5.11)$$

where $\hat{\eta}_1$ and $\hat{\eta}_2$ are variable directions of propagation of the first and second photons, and where $Y_{jm}{}^1$ is the spherical vector function for total angular momentum $j$, magnetic quantum number $m$, and parity $-1$ (see, for example, Akhiezer and Beretstetskii (1965) for notation). Now suppose that the lenses which make the photons impinge normally upon the polarisation analysers collect light in cones of half-angle $\xi$. The wavefunction of a photon pair which emerges from the pair of lenses can be represented as $D(\xi)\Psi$, where $D(\xi)$ is an operator which is exhibited in the appendix to Shimony (1971). An argument is outlined in that paper that if $\xi$ is infinitesimal, then $D(\xi)\Psi$ is equal (except for normalisation) to the ideal two-photon polarisation vector $\Psi_0$ of equation (5.7). This is a reasonable result, since there is no orbital angular momentum if the two photons propagate along a straight line. Therefore the fact that the photon pair has total angular momentum 0 implies that it has zero spin angular momentum, as in the state $\Psi_0$. Of course, a finite value of $\xi$ is essential in an actual experiment in order to obtain a non-vanishing count rate. The quantum-mechanical prediction for the coincidence rates with the polarisation state $D(\xi)\Psi$ is then:

$$[R(\phi)/R_0]_{D(\xi)\Psi} = \langle D(\xi)\Psi | Q(a) \otimes Q(b) | D(\xi)\Psi \rangle = \tfrac{1}{4} + \tfrac{1}{4} F_1(\xi) \cos(2\phi) \qquad (5.12)$$

where $F_1(\xi)$ is a monotonically decreasing function, has the value 1 for $\xi = 0$, and diminishes to 0·9876 at $\xi = 30°$. Equation (5.12) shows a somewhat weaker polarisation correlation than one finds in equation (5.9), as a result of the admixture of orbital angular momentum states when light is collected in a non-zero solid angle. However, the diminution of correlation is small, even for fairly large values of $\xi$, and it is evident that inequality (5.5) will be violated by the probabilities of equation (5.12), with $\xi = 30°$.

*5.2.3. Quantum-mechanical correlation for $J = 0 \to 1 \to 0$ cascade in an actual experiment.* In an actual experiment one does not have ideal linear polarisation analysers, and equation (5.12) must be corrected in order to take into account the inefficiency of actual analysers. We let $\epsilon_M{}^j$ be the maximum transmittance of the $j$th analyser ($j = 1, 2$) and $\epsilon_m{}^j$ be the minimum transmittance. (The former is 1 and the latter is 0 for an ideal analyser, but values for the analysers will be given in the summaries below of the experiments which have actually been performed.) Then equation (5.9) must be replaced by the following:

$$
\begin{aligned}
[R(a, b)/R_0]_{QM} ={}& \epsilon_M{}^1 \epsilon_M{}^2 \langle D(\xi)\Psi | Q(a) \otimes Q(b) | D(\xi)\Psi \rangle \\
&+ \epsilon_M{}^1 \epsilon_m{}^2 \langle D(\xi)\Psi | Q(a) \otimes \bar{Q}(b) | D(\xi)\Psi \rangle \\
&+ \epsilon_m{}^1 \epsilon_M{}^2 \langle D(\xi)\Psi | \bar{Q}(a) \otimes Q(b) | D(\xi)\Psi \rangle \\
&+ \epsilon_m{}^1 \epsilon_m{}^2 \langle D(\xi)\Psi | \bar{Q}(a) \otimes \bar{Q}(b) | D(\xi)\Psi \rangle
\end{aligned}
\tag{5.13}
$$

where

$$
\bar{Q}(a) = 1 - Q(a) \qquad \bar{Q}(b) = 1 - Q(b). \tag{5.14}
$$

We thus find:

$$
[R(\phi)/R_0]_{QM} = \tfrac{1}{4}(\epsilon_M{}^1 + \epsilon_m{}^1)(\epsilon_M{}^2 + \epsilon_m{}^2) + \tfrac{1}{4}(\epsilon_M{}^1 - \epsilon_m{}^1)(\epsilon_M{}^2 - \epsilon_m{}^2)F_1(\xi)\cos 2\phi
\tag{5.15}
$$

(see Clauser *et al* 1969, Horne 1970, Shimony 1971). Again, the quantum-mechanical counterpart of inequality (5.5) is violated, if suitable values of the transmittances are used.

*5.2.4. Other cascades.* If the photon pair is obtained from a $J = 1 \to J = 1 \to J = 0$ cascade with equal populations in the initial Zeeman sublevels and no coherence among them (so that the density matrix of the initial level is $\tfrac{1}{3}I$), but the preceding experimental arrangement is otherwise unchanged, then the quantum-mechanical prediction for the probability of joint detection is the same as the right-hand side of equation (5.15), except that $F_1(\xi)$ is replaced by $-F_2(\xi)$, where $F_2(0) = 1$. The function $F_2(\xi)$ decreases monotonically more rapidly than $F_1(\xi)$ (Clauser *et al* 1969, Horne 1970, Holt 1973). A systematic survey of other possible cascades has been made by Fry (1973).

*5.3. Description of experiments*

So far, there have been four experiments of the type just described. Three of these have agreed with the quantum-mechanical predictions, and one has agreed with the predictions by local realistic theories via Bell's theorem.

*5.3.1. Experiment by Freedman and Clauser (1972).* Freedman and Clauser (1972, see also Freedman 1972) observed the 5513 Å and 4227 Å pairs produced by the $4p^2\ ^1S_0 \to 4p4s\ ^1P_1 \to 4s^2\ ^1S_0$ cascade in calcium. Their arrangement is shown schematically in figure 5. Calcium atoms in a beam from an oven were excited by resonance absorption to the $3d4p\ ^1P_1$ level, from which a considerable fraction decayed to the $4p^2\ ^1S_0$ state at the top of the cascade. No precaution was necessary for eliminating isotopes with non-zero nuclear spin, since 99·855% of naturally occurring calcium has zero nuclear spin. Pile-of-plates polarisation analysers were used, with transmittances $\epsilon_M^1 = 0.97 \pm 0.01$, $\epsilon_m^1 = 0.038 \pm 0.004$, $\epsilon_M^2 = 0.96 \pm 0.01$, $\epsilon_m^2 = 0.037 \pm 0.004$. Each analyser could be rotated by angular increments of $\pi/8$, and the plates could be folded out of the optical path on hinged frames. The half-angle $\xi$ subtended by the primary lenses was 30°. Coincidence counting was done for 100 s periods; periods during which all plates were removed alternated with periods during which all were inserted. In each run the ratios $R(\pi/8)/R_0$ and $R(3\pi/8)/R_0$ were determined. Corrections were made for accidental coincidences, but even without this correction, the results still significantly violated inequality (5.6). The average ratios for roughly 200 h of running time are:

$$[R(\pi/8)/R_0]_{\mathrm{expt}} = 0.400 \pm 0.007 \qquad [R(3\pi/8)/R_0]_{\mathrm{expt}} = 0.100 \pm 0.003$$

and therefore:

$$[R(\pi/8)/R_0 - R(3\pi/8)/R_0]_{\mathrm{expt}} = 0.300 \pm 0.008$$

in clear disagreement with inequality (5.6). The quantum-mechanical predictions are obtained from equation (5.15) (with allowances for uncertainties in the measurement of the transmittances and the subtended angle):

$$[R(\pi/8)/R_0 - R(3\pi/8)/R_0]_{\mathrm{QM}} = (0.401 \pm 0.005) - (0.100 \pm 0.005) = 0.301 \pm 0.007.$$

The agreement between the experimental results with the quantum-mechanical predictions is excellent. Agreement is also found for other values of the angle $\phi$, as well as for measurements made with only one or the other polariser removed.

*5.3.2. Experiment by Holt and Pipkin (1973).* Holt and Pipkin (1973, see also Holt 1973) observed 5676 Å and 4047 Å photon pairs produced by the $9^1P_1 \to 7^3S_1 \to 6^3P_0$ cascade in the zero nuclear-spin isotope $^{198}Hg$ (see figure 6 for a partial level diagram of mercury). Atoms were excited to the $9^1P_1$ level by a 100 eV electron beam. The density matrix of the $9^1P_1$ level was found to be approximately $\frac{1}{3}I$ by measurements of the polarisation of the 5676 Å photons, so that equation (5.15) with $F_1(\xi)$ replaced by $-F_2(\xi)$ is used to calculate the quantum-mechanical predictions for the coincidence counting rates. Calcite prisms were employed as polarisation analysers, with measured transmittances:

$$\epsilon_M^1 = 0.910 \pm 0.001 \qquad \epsilon_M^2 = 0.880 \pm 0.001$$
$$\epsilon_m^1 < 10^{-4} \qquad \epsilon_m^2 < 10^{-4}.$$

The half-angle $\xi$ was taken to be 13° $(F_2(13°) = 0.9509)$. The quantum-mechanical prediction is:

$$[R(3\pi/8)/R_0 - R(\pi/8)/R_0]_{\mathrm{QM}} = 0.333 - 0.067 = 0.266$$

which only marginally exceeds the value $\frac{1}{4}$ allowed by inequality (5.6). The experimental result in 154·5 h of coincidence counting, however, is:

$$[R(3\pi/8)/R_0 - R(\pi/8)/R_0]_{\mathrm{expt}} = 0.316 \pm 0.011 - 0.099 \pm 0.009 = 0.216 \pm 0.013$$
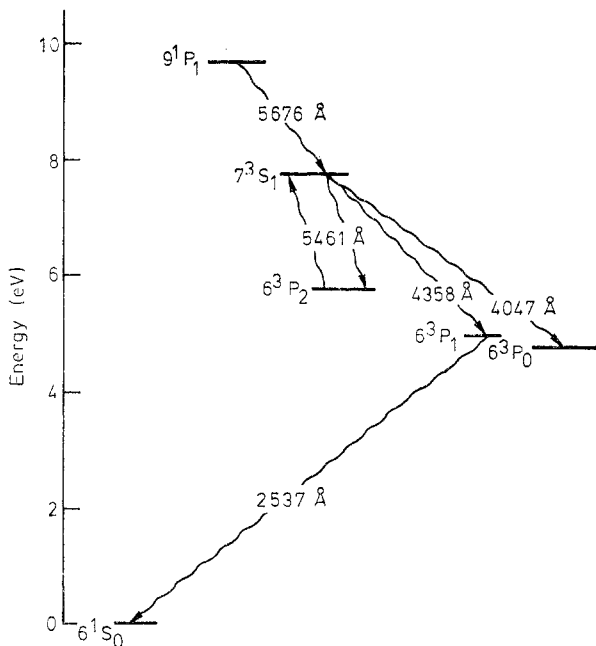
**Figure 6.** Partial level scheme for atomic mercury. Experiments by Holt and Pipkin, and by Clauser, excited the atoms to the $9^1P_1$ level by electron bombardment, and observed photons emitted by the $9^1P_1 \to 7^3S_1 \to 6^3P_0$ cascade. The experiment by Fry and Thompson excited atoms to the $6^3P_2$ (metastable) level by electron bombardment. Downstream, the atoms were excited by a tunable dye laser to the $7^3S_1$ level, and photons were observed from the $7^3S_1 \to 6^3P_1 \to 6^1S_0$ cascade.

in good agreement with inequality (5.6) but in sharp disagreement with the quantum-mechanical prediction. Since this result is very surprising, Holt and Pipkin took great care to check possible sources of systematic error: the contamination of the source by isotopes with non-zero nuclear spin, perturbation by external magnetic or electric fields, coherent multiple scattering of the photons (radiation trapping), polarisation sensitivity of the photomultipliers, and spurious counts from residual radioactivity and/or cosmic rays, etc.

    One such systematic error was found in the form of stresses in the walls of the Pyrex bulb used to contain the electron gun and mercury vapour. Estimates of the optical activity of these walls were then made, and the results were corrected correspondingly. (The values presented above include this correction.) It is noteworthy, however, that only the retardation sum for both windows was measured, for light entering the cell from one side and exiting through the opposite side. On the other hand, in the present experiment in which light exits from both windows, the relevant quantity is the retardation difference.

    It is also noteworthy that in the subsequent experiment by Clauser (§5.3.3), a correlation was first measured which agreed with the results of Holt and Pipkin. Stresses were then found in one lens which were due to an improper mounting. (These were too feeble to be detected by a simple visual check using crossed Polaroids.) The stresses were removed, the experiment was re-performed, and excellent agreement with quantum mechanics was then obtained. On the other hand, Holt and Pipkin did not repeat their experiment when they discovered the stresses in their bulb.

A second criticism is that Holt and Pipkin took the solid-angle limit to be that imposed by a field stop placed outside the collimating lenses. It is possible that lens aberrations may have allowed a larger solid angle than they recognised. A ray-tracing calculation was in fact performed to assure that this was not the case. However, a solid stop ahead of the lens would have given one greater confidence that this did not, in fact, occur.

*5.3.3. Experiment by Clauser (1976).* Clauser (1976) repeated the experiment of Holt and Pipkin, using the same cascade and same excitation mechanism, though with a source consisting mainly of the zero-spin isotope $^{202}$Hg. (The depolarisation effect due to some residual non-zero nuclear spin isotopes was calculated, using some results of Fry (1973).) Pile-of-plates polarisers were used with transmittances:

$$\epsilon_M^1 = 0.965 \qquad \epsilon_m^1 = 0.011 \qquad \epsilon_M^2 = 0.972 \qquad \epsilon_m^2 = 0.008$$

and the half-angle $\xi$ taken to be $18.6°$. The quantum-mechanical prediction is:

$$[R(3\pi/8)/R_0 - R(\pi/8)/R_0]_{QM} = 0.2841.$$

The experimental result, from 412 h of integration, is:

$$[R(3\pi/8)/R_0 - R(\pi/8)/R_0]_{expt} = 0.2885 \pm 0.0093$$

in excellent agreement with the quantum-mechanical prediction, but in sharp disagreement with inequality (5.6).

*5.3.4. Experiment by Fry and Thompson (1976).* Fry and Thompson (1976) observed the 4358 Å and 2537 Å photon pairs emitted by the $7^3S_1 \rightarrow 6^3P_1 \rightarrow 6^1S_0$ cascade in the zero nuclear-spin isotope $^{200}$Hg. Their experiment is shown schematically in figure 7.
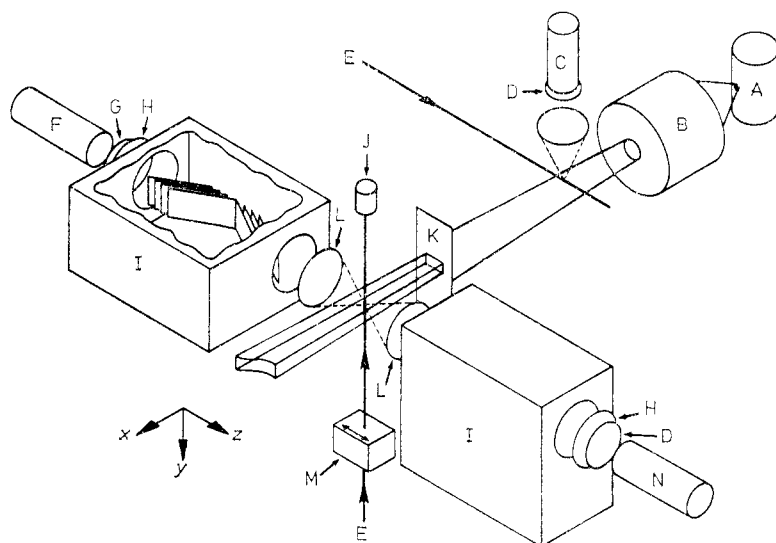
**Figure 7.** Schematic diagram of the experimental arrangement of Fry and Thompson. Polariser plate arrangement is also indicated. Actual polarisers have 14 plates. A, Hg oven; B, solenoid electron gun; C, RCA 8575; D, 4358 Å filter; E, 5461 Å laser beam; F, Amperex 56 DUVP/03; G, 2537 Å filter; H, focusing lens; I, pile-of-plates polariser; J, laser beam trap; K, atomic beam defining slit; L, light collecting lens; M, crystal polariser; N, RCA 8850 (figure after Fry and Thompson).

An atomic beam consisting of natural mercury was used as a source of ground-state ($6^1S_0$) atoms. The excitation of these to the $7^3S_1$ level occurred in two steps at different locations along the beam. First, the atoms were excited by electron bombardment to the metastable $6^3P_2$ level. Downstream, where all rapidly decaying states had vanished, a single isotope was excited to the $7^3S_1$ level by resonant absorption of 5461 Å radiation from a narrow-bandwidth tunable dye laser. The technique provided a high data accumulation rate, since only the cascade of interest was excited. Photons were collected over a half-angle $\xi$ of $19\cdot9° \pm 0\cdot3°$, and pile-of-plates analysers were used, with transmittances:

$$\epsilon_M^1 = 0\cdot98 \pm 0\cdot01 \qquad \epsilon_m^1 = 0\cdot02 \pm 0\cdot005 \qquad \epsilon_M^2 = 0\cdot97 \pm 0\cdot01 \qquad \epsilon_m^2 = 0\cdot02 \pm 0\cdot005.$$

The density matrix of the $7^3S_1$ level was ascertained by polarisation measurements of the 4358 Å photons; it was found to be diagonal even though the Zeeman sublevels were not equally populated. The quantum-mechanical prediction is:

$$[R(3\pi/8)/R_0 - R(\pi/8)/R_0]_{QM} = 0\cdot294 \pm 0\cdot007.$$

The experimental result is:

$$[R(3\pi/8)/R_0 - R(\pi/8)/R_0]_{expt} = 0\cdot296 \pm 0\cdot014$$

in excellent agreement with the quantum-mechanical prediction, but again in sharp disagreement with inequality (5.5). Because of the high pumping rate attainable with the dye laser, it was possible to gather the data in a remarkably short period of 80 min which, of course, diminished the probability of errors due to variations in the operation of the apparatus, and facilitated checking for systematic errors.

### 5.4. *Are the auxiliary assumptions for cascade-photon experiments necessary and reasonable?*

We have seen that the data from the cascade-photon experiments are *sufficient* to refute the whole family of local realistic theories, if either the CHSH or the CH auxiliary assumption is accepted. Both assumptions are very reasonable. Yet both are conceivably false. One may ask the question: are the experimental data, by themselves, sufficient to refute the theories? Alternatively, is at least some auxiliary assumption *necessary*? The answer was given by CH, who contrived a local hidden-variables model, the predictions of which agree exactly with those of quantum mechanics.

One may then ask how reasonable are these assumptions. In particular do they disagree with any known experimental data? A similar question may also be asked about the counter-example. It seems highly artificial, but are any of its implications experimentally testable?

### 5.4.1. *Critique of the CH and CHSH assumptions.*
The CHSH assumption (§5.1) is, given that a pair of photons emerges from the polarisers, the probability of their joint detection is independent of the polariser orientation $a$ and $b$. It may appear that the assumption can be established experimentally by measuring detection rates when a controlled flux of photons of known polarisation impinges on each detector. From the standpoint of local realistic theories, however, these measurements are irrelevant, since the distribution $\rho$ when the fluxes are thus controlled is almost certain to be different from that governing the ensemble in the correlation experiments. We thus

see no way of directly testing this assumption, and thus no experiments with which it disagrees.

It is noteworthy, however, that there exists an important hidden-variables theory—the semiclassical radiation theory—which correctly predicts a large body of atomic physics data, but which denies both the CHSH assumption as well as a presupposition of it. The presupposition is that one can speak unequivocally of a photon's passage or non-passage through the polarisation analysers. In the semiclassical radiation theory, however, a photon partially passes its respective polariser and departs with a reduced (classical) amplitude. Furthermore, this amplitude depends upon the polariser's orientation and thereby determines the probability of the photon's subsequent detection (in violation of the CHSH assumption that all photons have the same detection probability, independent of either polariser's orientation). Nonetheless, the predictions for this theory are consistent with those by inequality (5.2), and the theory is refuted by the cascade-photon experiments (Clauser 1972). Evidently an alternative assumption is possible which allows inequality (5.2) to constrain theories denying this presupposition.

Such an assumption was provided by CH. This assumption (§5.1) is that for every pair of emissions, the probability of a count with the polariser in place is less than or equal to the corresponding probability with the polariser removed. This assumption appears reasonable because the insertion of a polarisation analyser imposes an obstacle between the source of the emissions and the detector, and it is natural to believe that an obstacle cannot increase the probability of detection. To be sure, we know of situations in which the insertion of an additional optical element (apparently an obstacle) does increase the probability of detection, e.g. the insertion of a diagonally oriented linear polariser between two crossed polarisers. However, the situation appropriate to the CH assumption is quite different from the one just mentioned, since no polarising elements follow the inserted polarisers. Moreover, if the third polariser is a two-channel device, such as a Wollaston prism, the increased detection rate observed in one channel occurs at the expense of the detection rate in the ortho-gonal channel. The sum of the rates from both channels actually decreases when the second polariser is inserted. Correspondingly, if the third polariser is replaced by a polarisation-insensitive detector, in a closer parallel to the situation of the cascade-photon experiments, then the detection rate is always reduced when a polariser is inserted ahead of this detector.

These considerations, unfortunately, are by no means sufficient to prove the CH assumption, since these observations concern ensemble-average probabilities. The CH assumption requires that the probability be diminished upon the insertion of a polariser *for all* λ.

*5.4.2. The counter-example by Clauser and Horne.* Clauser and Horne (1974) produced a local hidden-variables model whose predictions agree exactly with those by quantum mechanics. In their model the rate at which photons jointly pass through the polarisation analysers is in agreement with Bell's inequalities, but the joint detection rate agrees with the quantum-mechanical predictions. The model requires that the detected photon pairs be selected in a very special manner from among those which pass through the analysers, and that those which have not passed through a polariser have a different detection probability from those which have. Although the selection is done entirely locally, it does have the appearance of being highly artificial and, indeed, almost conspiratorial against the experimenter.

The model applies only as long as the net detector efficiencies are smaller than a certain maximum value $p_{max}$ which depends on the analyser efficiencies $\epsilon_+$ and $\epsilon_-$: that is (using the notation of §4) when:

$$[\eta f]_{expt} \leqslant p_{max}. \tag{5.16}$$

With the conditions holding for the experiment by Freedman and Clauser, these values are $[\eta f]_{expt} \approx 0.004$ and $p_{max} \approx 0.4$. This comparison can be improved somewhat by reference to some experimental results by Clauser†. Inequality (5.16) then becomes $\eta_{expt} \leqslant p_{max}$. For the experiment by Freedman and Clauser the value $\eta_{expt} \approx 0.06$ holds.

Despite our caution concerning the CHSH and CH assumptions, we regard the experimental refutation which relies upon them to be compelling. It is striking that only a highly artificial model has so far been found which is local and yet yields quantum-mechanical detection rates in the cascade-photon experiments, and even this model can be excluded by rather modest improvements in the apparatus. There is also some hope for a theorem to the effect that any model consistent with the experimental data will have anomalous features as does the CH model.

## 6. Positronium annihilation and proton–proton scattering experiments

### 6.1. Historical background

Two experiments testing predictions based on Bell's theorem have been performed using the high-energy photons produced by positronium annihilation. The historical background of these experiments is interesting. Wu and Shaknov (1950) determined the parity of the ground state of positronium by a method suggested by Wheeler that consisted of measuring the polarisation correlation of $\gamma$ rays produced by positronium annihilation. The photons Compton-scattered, and two-photon coincidences were observed as a function of azimuthal scattering angles, $a$ and $b$. Two relative angles 0 and $\pi/2$ were employed. From the ratio of these two coincidence rates they were able to infer that the parity of the ground state is negative. Bohm and Aharonov (1957), with different motivations, showed that these data are explained by quantum mechanics if the polarisation state of the photon pair is assumed to be:

$$\Psi_1 = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \tag{6.1}$$

† Clauser (1974) noticed that the parameters of existing experimental results were inappropriate to determine whether or not transmission and reflection of a photon at a dielectric surface (similar to one of the surfaces in a pile-of-plates polariser) are, in fact, mutually exclusive possibilities. He thus performed an experiment which confirmed that they are. This behaviour is in marked contrast to that of the semiclassical radiation theory, in which a photon is simultaneously transmitted and reflected by the surface. He also performed a variation of this experiment (unpublished) in which the dielectric surface was replaced by a fine mesh mirror ($\approx 50\%$ transmission), and again photons were observed to be either transmitted or reflected but not both simultaneously. One can conclude from this result that, at least for the purposes of local realistic theories, the simultaneous emission of a photon into any two different solid-angle elements does not occur. Since the probabilities relevant to the CH counter-example are conditional upon the photons actually entering the collimator, it follows that the solid-angle parameter $f$ can be dropped for the purposes of inequality (5.16).

but are incompatible with the assumption that the ensemble of photon pairs can be described by a mixture of states, each of which is a product of two single-photon polarisation states. They therefore concluded that the data of Wu and Shaknov confirm the existence of states of two-particle systems which are 'non-separable', even though the particles are spatially remote from each other (see appendix 1).

Clauser *et al* (1969) investigated the possibility of using the arrangement of Wu and Shaknov, perhaps with some variation, for the further purpose of checking whether the observed frequencies can violate Bell's inequalities. It is, indeed, easy to show that if efficient linear polarisation analysers existed for 0·5 MeV photons (with transmittances $\epsilon_M - \epsilon_m$ greater than $\sim 0.83$), then the quantum-mechanical values for the coincidence rates, with the joint polarisation state given by equation (6.1), can violate the inequalities (3.12) or (3.20). Unfortunately no such analysers exist, and Compton scattering does no more than give a scattering distribution, described by the Klein–Nishina formula, that is dependent upon the direction of linear polarisation. They concluded that no variant of the Wu–Shaknov experiment can provide a test of the predictions based on Bell's theorem (see Horne 1970).

### 6.2. *The experiment by Kasday, Ullman and Wu*

It was argued by Kasday *et al* (1970, 1975, hereafter referred to as KUW; see also Kasday 1971) that such photon pairs can be used to test the predictions based on Bell's theorem if one accepts two auxiliary assumptions: (i) in principle, ideal linear polarisers can be constructed for high-energy photons; (ii) the results, which would be obtained in an experiment using ideal analysers, and those obtained in a Compton scattering experiment, are correctly related by quantum theory.

Their experimental arrangement (a variant of that of Wu and Shaknov) is shown schematically in figure 8. Positrons were emitted by a $^{64}$Cu source, stopped and
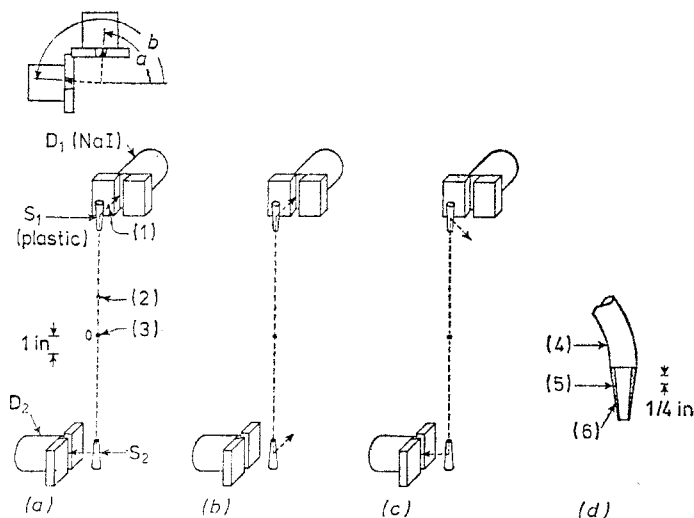


**Figure 8.** Schematic diagram of the experimental arrangement of KUW. The lead colli-
mator is not shown. (*a*) Four-fold coincidence event; (*b*) (*c*), three-fold coincidence
events; (*d*) detail of scatterer. *a*, *b* are the azimuthal angles of the scattered photons.
(1) Scattered $\gamma$ with energy $E$, absorbed by $D_1$, (2) annihilation $\gamma$, (3) positron source
and absorber, (4) light pipe, (5) plastic scatterer, (6) MgO-coated aluminium light
reflector (figure after Kasday *et al*).

annihilated in copper at the place labelled by 0. The annihilation $\gamma$-rays were emitted in all directions; the vertical direction was selected by a lead collimator. The scatterers were plastic scintillators. Lead slits selected a narrow range of acceptance azimuthal angle about the angles $a$ and $b$. The top slit-detector assembly was then rotated to vary the relative azimuthal angle. Accepted coincidence events had a four-fold coincidence among the two scatterers and two detectors, as well as a sum-energy requirement that the total energy deposited in each scatterer plus detector equals the annihilation energy. It is noteworthy that this is the only experiment which employs the arrangement of figure 1. (Here the ensemble consists of the pairs jointly scattered by the scintillators.)

KUW applied assumptions (i) and (ii) as follows. Imagine two ideal linear polarisation analysers in the plane perpendicular to the direction of propagation of the selected annihilation photons (the vertical direction in figure 8), which are respectively oriented in the directions $a$ and $b$ of the two slits. If the state $\lambda$ of the photon pair is given, a deterministic hidden-variables theory will determine whether each photon will pass through its respective analyser. This is their use of assumption (i). If photon 1 will pass its ideal analyser, then linear polarisation in the $a$ direction is assigned to it; and if photon 2 will correspondingly pass its analyser, then it is assigned linear polarisation in the direction $b$. KUW then use assumption (ii) to assert that the angular scattering distribution of each respective photon is given by the Klein–Nishina formula (a distribution which is dependent upon the photon's initial linear polarisation). With the Klein–Nishina formula one can calculate the probability that the scattered photons will enter the respective acceptance slits. Quantum mechanics makes a definite prediction for this joint probability. Deterministic local hidden-variables theories together with assumptions (i) and (ii) also imply an inequality governing this probability which will disagree with the quantum-mechanical predictions. The experimental data of KUW are in good quantitative agreement with the quantum-mechanical predictions.

This experiment is less decisive, in our opinion, as a refutation of the family of local realistic theories than are the cascade-photon experiments discussed in §5, because it relies upon assumptions which are considerably stronger than the assumption needed by the latter. If assumptions (i) and (ii) are not made, then a local hidden-variables model can be constructed (Horne 1970, Bell 1971 (see Kasday 1971)) which yields the same predictions for the experiment as those by quantum mechanics. This consideration, by itself, is not a fully sufficient reason to prefer the cascade-photon experiments, since a local hidden-variables model, albeit a much more artificial one, also exists which yields quantum-mechanical predictions for those experiments (see §5.4).

The relative strengths of the supplementary assumptions provides a better reason for preference. There is one respect in which assumption (ii) of KUW is quite unconvincing. The only definite polarisation states acknowledged by quantum theory are the various modes of elliptic polarisation (circular and linear polarisation being special cases of these). Since quantum theory can be used to calculate the relationship required for assumption (ii) between ideal and Compton polarimeters only when the state of a photon is one recognised by quantum theory itself, this assumption presupposes that photons which enter the Compton polarimeters are in a quantum-mechanically describable state. Such a supposition is strongly in conflict with the postulates of Bell's theorem. The state $\lambda$ presumably is not such a state, and moreover there is no prescription within quantum theory for calculating the results of an experiment for these more general states.

In a general hidden-variables scheme, the state $\lambda$ of the photons clearly cannot be represented as one of definite linear polarisation (the special case in which it can is the hypothesis studied by Furry). KUW's decomposition of the state $\lambda$ of a photon into linear and/or circular polarisation basis states is undoubtedly not possible in general.

Indeed, even in a quantum-mechanical treatment of the problem, the photons are acknowledged not to be in a state of definite polarisation. Quantum mechanically, neither photon's polarisation is in a definite state, but each is in what is known as an 'improper mixture' of such states (see d'Espagnat (1976) for a discussion of improper mixtures). In quantum theory, the only correct procedure for handling such systems is to perform calculations for the composite two-photon state. Thus we see that the 'marriage' between quantum mechanics and a general local realistic theory required by assumption (ii) results in a fatally incorrect handling of both theories.

### 6.3. The experiments by Faraci et al, Wilson et al and Bruno et al

An experiment very similar to that of KUW (but with $^{22}$Na as a source) was performed by Faraci *et al* (1974) with very different results. Their data disagree sharply with the quantum-mechanical predictions based upon the polarisation state of equation (6.1), and are at the extreme limit permitted by Bell's inequalities (given the assumptions of KUW). Their data also showed a variation in correlation strength which depends upon the source-to-scatterer distances. Since their paper is quite condensed, it is difficult to conjecture whether or not a systematic error is responsible for these results. KUW, however, present various criticisms of this work as well as a clarification of various misinterpretations of their own work by these authors.

Wilson *et al* (1976) repeated the experiment using $^{64}$Cu as a source. In contrast with the results of Faraci *et al*, they found complete agreement with the quantum-mechanical predictions, and no significant variation of the correlation strength when the scatterer positions were changed.

Bruno *et al* (1977) also repeated the experiment using $^{22}$Na as a source, but used alternatively Cu and Plexiglass as the annihilator. To discriminate against multiple scattering events they imposed a sum-energy restriction as did KUW and Wilson *et al* (but not Faraci *et al*), and also varied the scatterer sizes. Residual triplet-positronium contribution was ascertained by the use of the different annihilator materials. Again, no violation of the quantum-mechanical prediction was observed, for any of various source–scatterer distances.

### 6.4. Proton–proton scattering experiment by Lamehi-Rachti and Mittig

The only test of the predictions in Bell's theorem which has been performed so far not using photons is that of Lamehi-Rachti and Mittig (1976). They measured the spin correlations in proton pairs prepared by low-energy S-wave scattering. The scattering geometry is shown in figure 9. Protons from the Saclay tandem accelerator were scattered by a target containing hydrogen. The incident and recoil protons each entered analysers at $\theta_{lab} = 45°$ ($\theta_{cm} = 90°$). The protons were scattered by a carbon foil, and detected at positions labelled $L_1$ or $R_1$, and $L_2$ or $R_2$ in the figure. Coincidences were sought between detectors on opposite arms, as they varied the azimuthal angle of the detector pair of one arm.
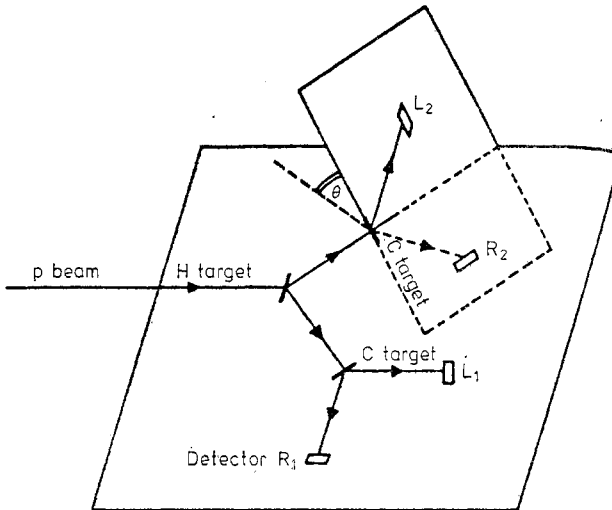
**Figure 9.** Proton–proton scattering geometry for the experiment by Lamehi-Rachti and Mittig (after Lamehi-Rachti and Mittig 1976).

Auxiliary assumptions similar to those required for the positronium experiments allow them to compare the data with the predictions for local realistic theories. It should be noted that their geometry requires an additional assumption not necessary for the positronium experiments. Since the analysers are only sensitive to the transverse components of the spin, and since $\theta_{cm} = 90°$, the correlation is of the form:

$$E(a, b) = C \cos a \cos b$$

and cannot violate inequalities (3.12) or (3.19) no matter what value $C \leqslant 1$ has. They thus assume that the quantum-mechanically predicted rotational invariance of the S-wave scattering (supposing negligible triplet contribution) allows them to decompose the correlation into a rotationally invariant part (singlet) and a non-rotationally invariant part (triplet). They then extrapolate the results back to a form which violates Bell's inequalities, and rely upon other experimental evidence to set an upper limit to the triplet scattering contribution. An arrangement in which one of the protons is electrostatically deflected through 90°, or magnetically precessed through 90°, would have eliminated the need for this last assumption.

They obtain good agreement with the quantum-mechanical predictions. If one accepts their assumptions, then Bell's inequalities are violated. However, even more reliance on quantum mechanics is needed than for the positronium experiments, and the criticisms of those experiments apply here more acutely.

## 7. Evaluation of the experimental results and prospects for future experiments

### 7.1. Two problems

There are two very different problems involved in evaluating the experiments so far performed for testing the predictions in Bell's theorem. The first is to determine the significance of the anomalous results of Holt and Pipkin and Faraci *et al.* The second is to determine what possibilities remain open if only the experiments which

favour quantum mechanics are accepted as veridical, and the anomalous results are attributed to spurious effects. In this section we discuss both of these problems.

*7.1.1. Significance of the experimental discrepancies.* The probability is extremely high, in our opinion, that the results contradicting the predictions by quantum mechanics were due to systematic errors. This opinion is *not* based on a conservative acknowledgment of the great success of quantum mechanics in the atomic domain. Rather, it is based upon the consideration that quantum mechanics predicts strong correlations, whereas Bell's theorem sets a limit upon such correlations. Virtually any conceivable systematic error will wash out a strong correlation so as to produce results in accordance with Bell's theorem, rather than speciously strengthen a weak correlation. We also note that the predictions by quantum mechanics are quantitatively precise. Therefore, in order to maintain that a local realistic theory governs nature, one must invoke experimental errors not only to explain a violation of the inequalities in seven out of nine experiments, but also to explain a very close quantitative agreement with the quantum-mechanical predictions in these seven. In view of the delicacy of these experiments we are not surprised that two anomalous results were obtained among nine. Experience with the experimental techniques and an awareness of the probable systematic errors, one expects, will lead to greater uniformity of results in later repetitions of the experiments. The results of the more recent experiments already indicate this to be so.

*7.1.2. Loopholes with auxiliary assumptions.* The assumptions for cascade-photon experiments are criticised in §5.4 and those for the positronium and the proton–proton scattering experiments in §§6.2 and 6.4. The opinion is advanced that those for the former are considerably weaker than those for the latter; hence, the cascade-photon experiments are to be preferred. Evidently, none of these assumptions can be directly tested, and thus neither argument is at present fully conclusive.

On the other hand, an indirect test of the assumptions of CH and CHSH may become possible. The counter-example for the cascade-photon experiments (in contrast to that for the positronium and the proton–proton scattering experiments) exploits minor technological imperfections in the apparatus. Indeed, improvements in the polariser efficiencies and/or the photomultiplier quantum efficiencies can make this counter-example obsolete. There is, to our knowledge, nothing fundamentally restricting significant improvements in either of these.

The cascade-photon experiments performed so far were all done on a very small budget (in comparison with modern large-scale experimentation). They were designed simply for testing inequality (5.2), and the various arrangements were sufficient to that end. Now suppose that a theorem (a strengthening of the one conjectured in §5.4) can be proved that the model of CH is essentially the only local hidden-variables model which reproduces quantum-mechanical data in the cascade-photon experiments. Since only a modest improvement in some of these parameters is sufficient to rule out this counter-example, the added expense of a significantly improved apparatus, in our opinion, would be justified.

*7.2. Experiments without auxiliary assumptions about detector efficiencies*

Even though the experimental results concerning local realistic theories appear highly convincing, it is still desirable to have an experiment for which auxiliary

assumptions are not required. It was shown in §4 that the requirements for such a scheme are demanding. Experiments using photons for this purpose do not appear to be feasible in the foreseeable future, since there seems to be no way of resolving the dilemma that highly efficient polarisation analysers can be achieved only for low-energy photons, while highly efficient detectors can be made only for high-energy photons. Furthermore, the two-body decay requirement is problematic with low-energy photons. Charged particles are evidently unusable, since an elegant argument by Bohr (see Mott and Massey 1965) indicates that magnetic state selection of their spin components violates the uncertainty principle. Hence, most schemes under consideration involve using either neutral particles and/or discrete states other than those associated with spin components. For example, Bell (1971) and Clauser *et al* (see Fehrs 1973) were inspired by a paper of Inglis (1961) to consider the charge-conjugation correlations shown in the decay of neutral kaon pairs produced by proton–antiproton annihilation. It was concluded that the exponentially decaying envelope of the correlations precludes the observation of a direct violation of Bell's inequalities in this system.

There is hope that the requirements for efficient analysis and detection can be achieved by observing the dissociation fragments of a metastable molecule, with a pair of Stern–Gerlach magnets as analysers. The latter have virtually 100% transmission, and proper design of the magnetic fields can minimise spurious spin-flips (Majorana transitions) during propagation of the decay fragments. Alkali metal and halogen atoms, if used as the decay fragments, can be detected individually by ionisation or electron attachment at a hot surface with nearly 100% efficiency. The parameters *a* and *b* can be taken to be the amplitudes of suitable resonant radio-frequency fields, applied in such a way as to coherently rotate the particle spins. Such an experiment holds promise of testing local realistic theories without any auxiliary assumptions, and with no loopholes other than the possibility of communication between the analysers (see §7.3).

### 7.3. Preventing communication between the analysers

Both the special and general theories of relativity preclude the existence of action-at-a-distance. This fact is, of course, the primary motivation for the various locality postulates considered above. However, in all of the experiments described so far, action-at-a-distance in the relativistic sense is not precluded, since the analysers are always kept at fixed orientations for periods of several seconds. Thus, there is ample time for information about the orientation of one analyser to be transmitted by some unknown mechanism (consistent with relativity theory) to the other apparatus (and/or other particle) thereby influencing its results. It is thus conceivable that such a mechanism is instrumental in producing quantum-mechanical coincidence counting rates in the above experiments. To test this possibility requires an experiment in which the parameters *a* and *b* are adjusted with great rapidity while the correlated particles are in flight. If the event consisting of the adjustment of the parameter *a* of the first analyser is wholly space-like separated from the detection event of particle 2, and similarly concerning adjustment of parameter *b* and the detection of particle 1, then no signal with subluminal speed can convey information about the orientation of one analyser to the other apparatus in time to affect the probability of detecting the respective particles. In other words, if the parameters *a* and *b* are adjusted with sufficient rapidity, then the non-occurrence of action-at-a-distance implies locality. For

photons the required rapid adjustment of the analyser orientations can be accomplished, for example, by using modern electro-optical devices such as high-speed Pockell's cells. Aspect (1976) proposed the use of acousto-optical devices for basically the same purpose.

However, even with such devices it is impossible to block the loophole completely. Since the backward light cones of the detection and adjustment events overlap, it may be claimed that events in the overlap region are responsible for determining the choices of the parameters *a* and *b* as well as the observed results. In this way the quantum-mechanical coincidence counting rates can still be accounted for without any direct causal connection between opposite sides of the experiment, and hence without introducing action-at-a-distance. Such an argument, however, seems unacceptable on methodological grounds, for it could be used to justify an *ad hoc* dismissal of any disagreeable data in almost any conceivable scientific experiment.

*7.4. Conclusion*

Although further experimental investigations of the family of theories governed by Bell's theorem are desirable, we are tentatively convinced that no theory of this kind can correctly describe the physical world. Nonetheless, we find this conclusion disturbing, since the philosophical point of view which most working scientists have found natural, at least until quite recently, requires a local realistic theory. Because of the evidence in favour of quantum mechanics from the experiments based upon Bell's theorem, we are forced either to abandon the strong version of EPR's criterion of reality—which is tantamount to abandoning a realistic view of the physical world (perhaps an unheard tree falling in the forest makes no sound after all)—or else to accept some kind of action-at-a-distance. Either option is radical, and a comprehensive study of their philosophical consequences remains to be made.

**Appendix 1. Criticism of EPR argument by Bohr, Furry and Schrödinger**

The argument of EPR is powerful, since their conclusion surely follows from their plausible premises. Most of the community of physicists rejected EPR's conclusion, however, because of a reply by Bohr (1935), which essentially consisted of a subtle analysis of their premise (ii). His argument is that when the phrase 'without in any way disturbing the system' is properly understood it is incorrect to say that system 2 is not disturbed by the experimentalist's option to measure *a* rather than *a'* on system 1.

'Of course there is, in a case like that just considered, no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially the question of *an influence on the very conditions which define the possible types of predictions regarding the future behaviour of the system*. Since these conditions constitute an inherent element of the description of any phenomenon to which the term "physical reality" can be properly attached, we see that the argumentation of the mentioned authors does not justify their conclusion that quantum-mechanical description is essentially incomplete.'

It is beyond the scope of the present review to analyse Bohr's claim that the term 'reality' can be used unambiguously in microphysics only when the experimental arrangement is specified. We are not convinced that Bohr ever succeeded in giving a

coherent statement of his philosophical position (see, for example, Shimony 1971, Stein 1972, Hooker 1972). We must admit, however, in consideration of the experimental evidence presented in this review against EPR's conclusion, that Bohr's position remains as one of the few feasible options concerning the foundations of quantum mechanics.

An early important reaction to the argument by EPR was to question premise (i). Furry (1936) and Schrödinger (1935) independently considered the possibility that, after systems 1 and 2 become spatially separated and cease effectively to interact, their joint wavefunction no longer has the form (2.1), but rather becomes a mixture of simple product states, each having the form:

$$\Psi_{\hat{n}} = u_{\hat{n}}{}^{\pm}(1) \otimes u_{\hat{n}}{}^{\mp}(2). \tag{A1.1}$$

For each element of the mixture, both 1 and 2 are then in definite quantum states. This possibility is sometimes called 'Furry's hypothesis', but that nomenclature is inappropriate. What Furry did was to show that for any choice of $u_{\hat{n}}(1)$ and $u_{\hat{n}}(2)$, there exist in principle pairs of observables, $M$ of 1 and $S$ of 2, such that the statistical predictions for joint measurements of $M$ and $S$ based upon mixtures of the $\Psi_{\hat{n}}$ are different from those based upon equation (2.1). Since Furry believed quantum mechanics to be correct, he concluded that a state like that of equation (2.1) does not automatically evolve into a mixture of the $\Psi_{\hat{n}}$ when 1 and 2 separate from each other (see the conclusion of §4 of his paper). It is more appropriate to call this scheme 'Schrödinger's hypothesis', since he explicitly stated that it may be true. Strong evidence against this hypothesis was presented in 1957 by Bohm and Aharonov (see §6.1). More recent experimental evidence confirming their conclusions has been discussed by Kasday (1971) and Clauser (1972, 1977). It is noteworthy that this hypothesis is such a natural one that many physicists apparently believe it to be a resolution of the EPR 'paradox' without recognising the theoretical and experimental evidence against it.

## Appendix 2. Hidden-variables theories

A theory which asserts that the quantum-mechanical description of a physical system is incomplete and requires supplementation in order to specify completely the state of the system is commonly called a *hidden-variables theory*†. Those properties of the system which are proposed as supplements to the quantum-mechanical description are commonly called *hidden variables* or sometimes *hidden parameters*. The history of hidden-variables theories is quite intricate, because the various proponents and opponents have made different assumptions about the conditions of adequacy which a hidden-variables theory should satisfy. We shall review here only as much of this history as is needed to provide the background for Bell's theorem. Other reviews of this subject (some including discussions of Bell's theorem) may be found in Bell (1966), Capasso *et al* (1970), Belinfante (1973), Jammer (1974) and d'Espagnat (1976).

In 1926–7 deBroglie wrote several papers proposing an interpretation of the wavefunction very different from that of Bohr. He supposed that the wavefunction

† The literature is not always consistent on this point, and many authors have included a hypothesis of determinism in their definition of a hidden-variables theory. In this review hidden-variables theories for which determinism holds are referred to as deterministic hidden-variables theories.

associated with a particle is a physically real field propagating in physical space in accordance with the Schrödinger equation. He also supposed that the particle always has a definite position and a definite momentum. Thus his interpretation was actually a hidden-variables theory. Finally, he assumed an intimate coupling between the particle and the field described by the function $\Psi$, so that the latter can be considered a 'guiding wave' or 'pilot wave' for the particle. This coupling then accounts for interference and diffraction phenomena. Several serious difficulties were found in deBroglie's theory (see deBroglie 1960), especially concerning many-particle systems and the S-wave state of a particle. As a result, deBroglie set aside his investigations of this kind until he was re-encouraged by the work of Bohm in 1952.

We shall not discuss in detail the various other models considered so far (see, for example, Madelüng 1926, deBroglie 1953, Bohm and Vigier 1954, Freistadt 1957, Andrade e Silva and Lochak 1969). Some of these models assume that in addition to the potential recognised in classical mechanics the particle is subject to a 'quantum potential' $h^2\nabla^2 R/2mR$, where $R$ is the amplitude of the wavefunction. Other models assume that the wavefunction describes an averaged or smoothed state of a fluid medium, subject to random fluctuations which are not taken into account by the wavefunction, but which are nevertheless important for understanding the statistical behaviour of particles moving in the medium. For the most part, the advocates of these models do not claim that they are anything but tentative descriptions of the subquantum level of the physical world. Their significance lies in providing existence proofs that a theory can be deterministic in character, and nevertheless agree with many of the statistical predictions by quantum mechanics. We have found, in our discussion of Bell's theorem in §3, that a theory can achieve complete agreement with quantum mechanics only if it is non-local.

Leaving aside the locality problem, we may ask how is it possible that so many mathematically consistent hidden-variables theories have been devised when various theorems have claimed that the structure of the class of quantum-mechanical observables precludes such theories? We now discuss two such theorems.

## A2.1. Von Neumann's theorem

The most famous theorem of this type is due to von Neumann (1932). Let $\mathcal{O}$ be the class of observables, and suppose that every self-adjoint operator on a Hilbert space $\mathcal{H}$ of dimension greater than 1 represents a member of $\mathcal{O}$ (but it is not excluded that $\mathcal{O}$ has other members). A state is specified by defining an expectation value Exp($A$) on every $A \in \mathcal{O}$, and it is assumed that Exp satisfies the following conditions.

(i) Exp(1) = 1, where 1 is the observable which, by definition, always has the value unity.

(ii) For each $A \in \mathcal{O}$ and each real number $r$, Exp($rA$) = $r$Exp($A$).

(iii) If $A$ is non-negative, then Exp($A$) $\geqslant 0$.

(iv) If $A, B, C, \ldots$, are arbitrary observables, then there is an observable $A + B + C + \ldots$ (which does not depend upon the choice Exp) such that:

$$\text{Exp}(A + B + C + \ldots) = \text{Exp}(A) + \text{Exp}(B) + \text{Exp}(C) + \ldots.$$

The theorem asserts that there exists a self-adjoint operator $A$ on $\mathcal{H}$ such that Exp($A^2$) $\neq$ [Exp($A$)]², i.e. the state defined by Exp is not dispersion-free over the quantum-mechanical observables.

The intuitive meaning of the conclusion of von Neumann's theorem is that no state

of the system—not even a state different from those recognised by quantum mechanics —can assign definite values simultaneously to all quantum-mechanical observables. Von Neumann's theorem is mathematically correct, but its physical significance is doubtful. The fact that it was often cited over three decades as a proof for the completeness of quantum mechanics is a kind of historical aberration, and has wrought much confusion. Its crucial weakness is the supposition that any possible state of the system must satisfy condition (iv), even when $A, B, C, \ldots$, are non-commuting operators and therefore represent observables which, according to quantum mechanics, cannot be simultaneously measured. The actual procedure for measuring $A + B$, when $A$ and $B$ do not commute, is different from the procedures for measuring $A$ and $B$ separately and does not presuppose any information about the value of either $A$ or $B$. Consequently, the fact that the additivity of condition (iv) is satisfied by quantum-mechanical states is a peculiarity of quantum mechanics, and there is no reason to suppose that it is satisfied by non-quantum-mechanical states. This criticism of the physical significance of von Neumann's theorem was made by Siegel (1966) and Bell (1966). (In the two-dimensional Hilbert space of a spin-$\frac{1}{2}$ particle, Bell also constructed a family of dispersion-free states which are physically reasonable, even though they violate condition (iv). Pearle (1965) and Kochen and Specker (1967) independently constructed similar models.) There is evidence† that Einstein was critical of condition (iv) as early as 1938 and therefore did not consider von Neumann's theorem to be an obstacle to the 'completion' of quantum mechanics as demanded by the argument of EPR. An interesting survey by Jammer (1974, pp272–7) shows that others were critical of condition (iv), but not with complete clarity.

## A2.2. Gleason's theorem

In 1957 Gleason proved a theorem which is free from the unphysical condition (iv) of von Neumann's theorem, and which has frequently been considered to be a decisive proof of the impossibility of any consistent hidden-variables theory. We shall not state the theorem itself but rather shall state a corollary‡ which can be compared directly with von Neumann's theorem.

Let $\mathcal{O}$ be a class of observables containing all those represented by the self-adjoint operators on a Hilbert space of dimension greater than 2, and let Exp be a real-valued function over $\mathcal{O}$, which satisfies the following conditions:

1, 2, 3, as in von Neumann's theorem.

4′. If $A, B, C, \ldots$, are commuting self-adjoint operators on $\mathcal{H}$, then

$$\mathrm{Exp}(A + B + C + \ldots) = \mathrm{Exp}(A) + \mathrm{Exp}(B) + \mathrm{Exp}(C) + \ldots.$$

Then Exp is not dispersion-free.

This corollary is weaker than von Neumann's theorem in one respect: it does not apply to a Hilbert space of dimension 2, and therefore it permits the models of Bell, Kochen and Specker, and Pearle. But it is much stronger in one crucial respect: it requires additivity only over commuting operators, for which the values of $A + B + C +$

† Professor P G Bergmann was an assistant to Einstein at that time, and he reported Einstein's criticism to one of us. We regret that we have no evidence concerning Einstein's opinion of von Neumann's argument in 1935, when the paper of EPR was written.

‡ There are several direct proofs of essentially this corollary which do not rely upon the main theorem of Gleason: Bell (1966), Kochen and Specker (1967) and Belinfante (1973). The proof given by Jammer (1974, pp298–9) is the same as that of Bell, who is not credited.

. . . can in principle be determined by summing the results of simultaneous measurements of $A, B, C, \ldots,$.

The conditions for this corollary are physically plausible, and its conclusion seems to be strong enough to preclude all non-trivial hidden-variables theories (i.e. all which apply to a system which is quantum-mechanically represented by a Hilbert space of dimension greater than 2). However, Bell (1966) pointed out the possibility of a family of non-trivial hidden-variables theories which do not satisfy all the conditions of the corollary and therefore are not bound by its conclusion. Suppose $A$, $B$ and $C$ are self-adjoint operators such that $A$ commutes with both $B$ and $C$, but $B$ and $C$ do not commute with each other. Therefore $A$ can in principle be measured simultaneously with $B$, or it can be measured simultaneously with $C$, and different experimental arrangements are required for the two measurements. In the corollary it was assumed that when the state of the system is fully specified, the function $\text{Exp}(X)$ has a definite value for each observable $X$, however $X$ is measured. 'It was tacitly assumed that measurement of an observable must yield the same value independently of what other measurements may be made simultaneously' (Bell 1966, p451). But there is no *a priori* reason that this assumption should be true. 'The result of an observation may reasonably depend not only on the state of the system (including hidden variables) but also on the complete description of the apparatus' (Bell 1966, p451). It is physically reasonable, therefore, to consider hidden-variables theories in which the expectation values have the form $\text{Exp}(X; \mathscr{C})$, where $\mathscr{C}$ indicates the 'context' of the measurement of $X$, i.e. all the quantities measured simultaneously with $X$. Gleason's theorem and its corollary do not preclude the possibility that such a 'contextual' hidden-variables theory can be dispersion-free, so that the result of measuring any observable is precisely determined by the state of the system (including hidden variables) together with the 'context' of the measurement.

Bell's proposal of a new family of hidden-variables theories sheds light on models like the one given by Bohm in 1952. This model antedated Gleason's work, but Bohm (1952, p187) defended it against von Neumann's impossibility theorem in the following way:

'the so-called "observables" are . . . not properties belonging to the observed system alone, but instead potentialities whose precise development depends just as much on the observing apparatus as on the observed system. In fact, when we measure the momentum "observable", the final result is determined by hidden parameters in the momentum-measuring device as well as by hidden parameters in the observed electron.'

This passage does not propose the consideration of contextual hidden-variables theories as explicitly as Bell does, but it can be construed retrospectively as implicitly agreeing with Bell.

The next step in the history of hidden-variables theories was taken by Bell, once he was convinced that impossibility theorems like that of Gleason do not establish *a priori* the inconsistency of hidden-variables models. By taking these models seriously, he was free to examine whether they shared any physically unreasonable properties in spite of their mathematical consistency, and to inquire whether such properties are inevitable in any hidden-variables theory which agrees with the predictions by quantum mechanics. In this way he was heuristically led to the study of locality.

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