Solutions of HW #4

Problem #1 B & J 6.6

(a) $H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 - \frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{e}{m}\vec{p}\cdot\vec{A} + \frac{e^2}{2m}\vec{A}^2$

e.g. B&J (4.24)

(this is the Hamiltonian we used in describing interaction with EM wave)

Now: Coulomb gauge $\Rightarrow \vec{A} = \frac{\vec{B}\times\vec{r}}{2} \Rightarrow$

$$\vec{p}\cdot\vec{A} = \vec{p}\cdot(\vec{B}\times\vec{r})\frac{1}{2} = \frac{1}{2}\underbrace{(\vec{r}\times\vec{p})}_{\vec{L}}\cdot\vec{B}$$

$$\vec{A}^2 = \frac{1}{4}(\vec{B}\times\vec{r})^2 = \frac{1}{4}B^2\underbrace{r^2\sin^2\theta}_{x^2+y^2}$$

So,

$$H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 - \frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{e}{2m}\vec{L}\cdot\vec{B} + \frac{e^2}{8m}B^2(x^2+y^2)$$

Note that original H doesn't take into account interaction of $\vec{B}$ with spin $\Rightarrow$

$$\frac{e}{2m}\vec{L}\cdot\vec{B} \Rightarrow \frac{e}{2m}(\vec{L}+2\vec{S})\cdot\vec{B} = \frac{\mu_B}{\hbar}(\vec{L}+2\vec{S})\cdot\vec{B}$$

Then, $H = -\frac{\hbar^2}{2m}\vec{D}^2 - \frac{Ze^2}{4\pi\varepsilon_0 r} +$

$$+ \frac{\mu_B}{\hbar}\underbrace{(\vec{L}+2\vec{S})\cdot\vec{B}}_{\substack{\shortparallel \\ (L_z+2S_z)B \text{ if } \vec{B}\parallel Oz}} + \left(\frac{\mu_B}{\hbar}\right)^2\cdot\frac{m}{2}B^2(x^2+y^2) =$$

discard in strong $\vec{B}$

$$= -\frac{\hbar^2}{2m}\left(\underbrace{D_x^2+D_y^2}+\underline{D_z^2}\right) - \frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{\mu_B}{\hbar}\cdot$$

$$\cdot(L_z\underline{+2S_z})B + \frac{m}{2}\left(\frac{\mu_B}{\hbar}\right)^2\underbrace{(x^2+y^2)}B^2 \Rightarrow$$

$$\Rightarrow H = \underbrace{-\frac{\hbar^2}{2m}D_z^2 + \frac{\mu_B}{\hbar}(L_z+2S_z)B}_{\shortparallel\, H_z} +$$

$$+ \underbrace{\left(-\frac{\hbar^2}{2m}(D_x^2+D_y^2) + \frac{m}{2}\underbrace{\left(\frac{\mu_B B}{\hbar}\right)^2}_{\shortparallel\, \omega_L^2}(x^2+y^2)\right)}_{\shortparallel\, H_{xy}\ \leftarrow\ 2D\ H.O.}$$

So $E = \hbar\omega_L(m_\ell+2m_s) + \hbar\omega_L(n_x+n_y+1) + \frac{\hbar^2k^2}{2m} =$

$= \hbar\omega_L(2r+2m_s+1)$, $r = 0, 1, 2, \ldots$

$\underline{\Delta E = 2\hbar\omega_L}$

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(c) neutron stars $\Rightarrow$ $B \sim 10^8\,T$

$$\Delta E = 2\hbar\omega_L = 2\hbar \frac{\mu_B}{\hbar} B = 2\underbrace{\mu_B}_{9{,}27\cdot 10^{-24}\,\frac{J}{T}} \overbrace{B}^{10^8\,T} = 1{,}8\cdot 10^{-15}\,J =$$

$$\approx 10^4\,eV$$

huge!

Size $\sim \sqrt{\langle x^2+y^2\rangle}$ $\quad x = \sqrt{\frac{\hbar}{2m\omega_L}}\,(a+a^+)$

Recall: $\quad y = \sqrt{\frac{\hbar}{2m\omega_L}}\,(b+b^+)$

$$\langle n|x^2|n\rangle = \frac{\hbar}{2m\omega_L}\langle(a^2+a^{+2}+\underbrace{aa^+}_{\hat{N}+1}+\underbrace{a^+a}_{\hat{N}})\rangle =$$

$$= \frac{\hbar}{2m\omega_L}(2N+1) = \frac{\hbar}{2m\omega_L} = \langle y^2\rangle$$

ground state: $N=0$

$\hat{N}|N\rangle = N|N\rangle$

operator

So, size $\sim \sqrt{\frac{\hbar}{m\omega_L}} = \sqrt{\frac{\hbar^2}{m\cdot\underbrace{\mu_B}_{\frac{e\hbar}{2m}}B}} = \frac{\hbar\sqrt{2}}{\sqrt{e\hbar B}} = \sqrt{\frac{2\hbar}{eB}} \sim 3{,}5\cdot 10^{-12}\,m$

tiny!

much tighter confinement compared to Coulomb interaction

Problem # 2

$$\Psi(t=0) = \Psi_{n_1 l_1 m_{l_1}}(\vec{r}_1)\,\Psi_{n_2 l_2 m_{l_2}}(\vec{r}_2) = \frac{1}{\sqrt{2}}\Big(\Psi_{S\,(\text{or}\,+)}(\vec{r}_1,\vec{r}_2) + \Psi_{A\,(\text{or}\,-)}(\vec{r}_1,\vec{r}_2)\Big)$$

energy $E_+ = \underbrace{E^{(0)}}_{E_{n_1}+E_{n_2}} + J + K$

$E_- = E^{(0)} + J - K$

Then, $\Psi(t) = \frac{1}{\sqrt{2}}\Big(\Psi_+(t) + \Psi_-(t)\Big) =$

$$= \frac{1}{\sqrt{2}}\Big(\underbrace{\Psi_+(t=0)}_{\frac{1}{\sqrt{2}}(\psi_1(\vec{r}_1)\psi_2(\vec{r}_2)+\psi_2(\vec{r}_1)\psi_1(\vec{r}_2))} e^{-\frac{i}{\hbar}E_+ t} + \underbrace{\Psi_-(t=0)}_{\frac{1}{\sqrt{2}}(\psi_1(\vec{r}_1)\psi_2(\vec{r}_2)-\psi_2(\vec{r}_1)\psi_1(\vec{r}_2))} e^{-\frac{i}{\hbar}E_- t}\Big) =$$

$\psi_1 \leftarrow n_1 l_1 m_{l_1}$, $\psi_2 \leftarrow n_2 l_2 m_{l_2}$

$$= \frac{1}{2}\Big[\big(\psi_1(\vec{r}_1)\psi_2(\vec{r}_2) + \psi_2(\vec{r}_1)\psi_1(\vec{r}_2)\big)e^{-\frac{i}{\hbar}Kt} + \big(\psi_1(\vec{r}_1)\psi_2(\vec{r}_2) - \psi_2(\vec{r}_1)\psi_1(\vec{r}_2)\big)e^{\frac{i}{\hbar}Kt}\Big]e^{-\frac{i}{\hbar}(E^{(0)}+J)t} \;(=)$$

$$\ominus \; \frac{1}{2}\left[2\,\psi_1(\vec{r}_1)\psi_2(\vec{r}_2)\cos\frac{K}{\hbar}t - 2i\,\psi_2(\vec{r}_1)\psi_1(\vec{r}_2)\sin\frac{K}{\hbar}t\right] \cdot e^{-\frac{i}{\hbar}(E^{(0)}+J)t}$$

So, the particle 2 will be in $n_1 l_1 m_{l_1}$ state
1 $n_2 l_2 m_{l_2}$

when $\cos\frac{K}{\hbar}t = 0$ and $\sin\frac{K}{\hbar}t = 1$, i.e.
at $t = \frac{\pi\hbar}{2K}$ $(+\pi n)$ the occupation will be reversed
due to <u>exchange</u> interaction