PH451/551
QM

Lecture # 24

①

Time-dependent perturbation theory

Known:

$$H_0 |n\rangle = E_n |n\rangle$$

$$|\Psi(t=0)\rangle = \sum_n C_n |n\rangle$$

$$|\Psi(t)\rangle = \sum_n C_n e^{-\frac{i}{\hbar}E_n t} |n\rangle$$

Now: $H = H_0 + H'(t)$ => need to solve $H|\Psi\rangle = i\hbar \frac{d|\Psi\rangle}{dt}$

⇓

use time-dep. perturbation theory!

⇓

assume $|\Psi(t)\rangle = \sum_n \underline{\underline{C_n(t)}}\, e^{-\frac{i}{\hbar}E_n t} |n\rangle$

$H' \neq 0$ ↗

↖ find $C_n(t)$!

$$\left(H_0 + H'(t)\right) \underbrace{|\Psi(t)\rangle}_{= \sum_n C_n(t) e^{-\frac{i}{\hbar}E_n t} |n\rangle} = i\hbar \frac{d}{dt} \underbrace{|\Psi(t)\rangle}_{= \sum_n C_n(t) e^{-\frac{i}{\hbar}E_n t} |n\rangle}$$

LHS: $\sum_n \Big[E_n C_n(t) e^{-\frac{i}{\hbar}E_n t} |n\rangle + H'(t) C_n(t) \cdot e^{-\frac{i}{\hbar}E_n t} |n\rangle \Big] = i\hbar \sum_n \Big[\frac{dC_n}{dt} e^{-\frac{i}{\hbar}E_n t} |n\rangle - \frac{i}{\hbar} E_n C_n(t) \cdot e^{-\frac{i}{\hbar}E_n t} |n\rangle \Big]$

↑ RHS

which simplifies to

$$\sum_n H'(t)\, c_n(t)\, e^{-\frac{i}{\hbar}E_n t} |n\rangle = i\hbar \sum_n \frac{dc_n(t)}{dt} \cdot e^{-\frac{i}{\hbar}E_n t} |n\rangle$$

Now multiply by $\langle k|$:

RHS:

$$i\hbar \sum_n \frac{dc_n(t)}{dt} e^{-\frac{i}{\hbar}E_n t} \underbrace{\langle k|n\rangle}_{=\delta_{kn}} = i\hbar \frac{dc_k}{dt} e^{-\frac{i}{\hbar}E_k t}$$

LHS:

$$\sum_n \langle k| H' c_n(t) e^{-\frac{i}{\hbar}E_n t} |n\rangle = \sum_n c_n(t) \cdot e^{-\frac{i}{\hbar}E_n t} \underbrace{\langle k|H'|n\rangle}_{=H'_{kn}(t)}$$

$H'_{kn}(t) \leftarrow$ matrix element

Putting everything together:

$$i\hbar \frac{dc_k}{dt} = \sum_n c_n(t)\, e^{-\frac{i}{\hbar}(E_n - E_k)t}\, H'_{kn}(t)$$

Can be solved exactly for a two-level system, but difficult in a general case $\Rightarrow$ use perturbation series expansion $\Rightarrow$

- Present C_n as $C_n = \underbrace{C_n^{(0)}}_{} + \underbrace{C_n^{(1)}}_{\lambda} + \underbrace{C_n^{(2)}}_{\lambda^2} + \ldots$ (3)
- Collect terms of the same perturbation order (note $H = H_0 + \underline{\lambda} H'(t)$)

λ^0: $i\hbar \dfrac{dC_k^{(0)}}{dt} = 0 \Rightarrow C_k^{(0)} = \text{const} \leftarrow$ determine from initial conditions

λ^1: $i\hbar \dfrac{dC_k^{(1)}}{dt} = \sum_n C_n^{(0)}(t) e^{-\frac{i}{\hbar}(E_n - E_k)t} H'_{kn}(t)$

$\vdots$

Assume that the initial state at $t=0$ is

If $|\Psi(0)\rangle = |i\rangle \Leftarrow$ some $\to |i\rangle$ eigenstate of H_0

$\Downarrow$

$C_n^{(0)} = \delta_{ni} \Rightarrow$

$i\hbar \dfrac{dC_k^{(1)}}{dt} = e^{-\frac{i}{\hbar}(E_i - E_k)t} H'_{ki}(t) \Rightarrow$

$$C_k^{(1)} = \frac{1}{i\hbar}\int_0^t \langle k|H'(t')|i\rangle e^{\frac{i}{\hbar}(E_k - E_i)t'} dt'$$

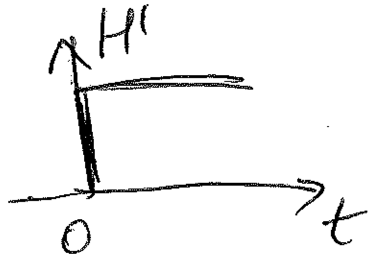
Probability of making transition from $|i\rangle$ to some state $|k\rangle$ is $|C_k^{(1)}|^2$

$$\mathcal{P}_{i\to f}(t) = \frac{1}{\hbar^2}\left|\int_0^t \langle f|H'(t')|i\rangle e^{i\omega_{fi}t'}\,dt'\right|^2 \qquad (4)$$

i – initial, f – final

most time-dep. interactions with external fields (including light abs. /emission) are described by this expression

Ex.



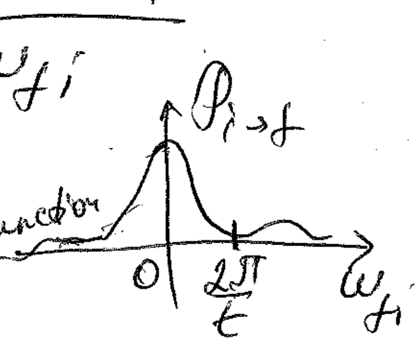
constant perturbation (in time)

$\mathcal{P}_{i\to f} = ?$

$$C_f(t) = \frac{1}{i\hbar}\langle f|H'|i\rangle \underbrace{\int_0^t e^{i\omega_{fi}t'}\,dt'}_{\displaystyle \text{"} \frac{e^{i\omega_{fi}t}-1}{i\omega_{fi}}} =$$

$$= \underbrace{\frac{2}{i\hbar}\langle f|H'|i\rangle e^{\frac{i\omega_{fi}t}{2}}}\cdot \underbrace{\frac{\sin\frac{\omega_{fi}t}{2}}{\omega_{fi}}}$$

as $t\uparrow \Rightarrow$ $\text{sinc}^2 \to \delta$-function



$$\mathcal{P}_{i\to f} = |C_f(t)|^2 = \frac{4|\langle f|H'|i\rangle|^2}{\hbar^2\omega_{fi}^2}\sin^2\frac{\omega_{fi}t}{2} =$$

$$= \frac{|\langle f|H'|i\rangle|^2}{\hbar^2}t^2\,\text{sinc}^2\frac{\omega_{fi}t}{2}$$