# Identical particles

Two particles are identical if all their intrinsic properties (mass, spin, charge, ...) are exactly the same, and the particles behave exactly the same way under equal physical conditions.

Examples of identical particles: 2 electrons, 2 protons, ...

Examples of non-identical particles: electron & position (differ by charge),
(distinguishable)

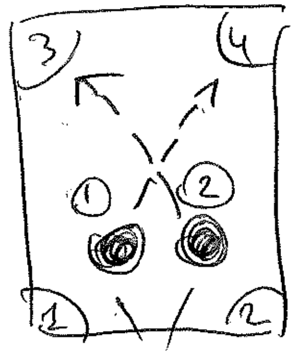
electron & proton (by charge & mass), electron & muon (mass),

What's the difference in treating identical particles in classical & quantum mechanics?

classical ⇓ well-defined trajectory ⇒ easy to follow & distinguish

quantum ⇓ no trajectories, just probabilities → no basis to distinguish

# Ex. 1 Billiard table




$t=0$

Two outcomes:

① → hole 3
② → hole 4

or

① → hole 4
② → hole 3

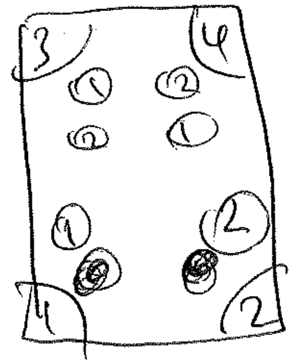
Hit the ball & see where it goes

⇓ trace the trajectories

distinguishable outcomes

⇓

CM case (classical mech.)




enter the room at t=τ

balls are already in ⇐ enter the room at t=τ

⇓

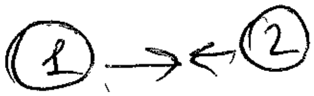
indistinguishable outcomes

⇓

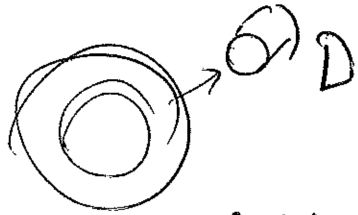
QM case

## Ex. 2 Collision of two identical particles





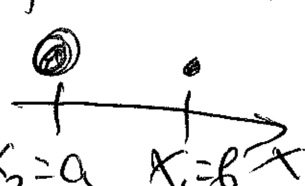

before collision
(two separate wavepackets)




after collision
(overlapped wavepackets)
$\Downarrow$
impossible to tell whether detector D sees particle 1 or 2
$\Downarrow$
exchange degeneracy
$\Downarrow$

**Ex.** Suppose we have a system of two distinguishable particles 1 & 2 & a position measurement on a system reveals particle 1 at $x = a$ & particle b at $x = b$ $\Rightarrow$ the state after the measurement is $|\psi\rangle = |x_1 = a, x_2 = b\rangle = |a, b\rangle$


If the other way around $\Rightarrow$




$\Rightarrow |\psi\rangle = |x_1 = b, x_2 = a\rangle = |b, a\rangle$

these are distinguishable outcomes!
What if the particles are identical? $|ab\rangle$? $|ba\rangle$?
$\Rightarrow$

combination of both!

$\Downarrow$

which combination is determined by the nature of the particles (bosons or fermions)

$\Downarrow$

Introduce the <u>exchange</u>, or <u>permutation</u> operator $\Rightarrow$ $\hat{P}_{ij}$

For $N$ particles $\Rightarrow$

$$\hat{P}_{ij}\,\Psi(\vec{r}_1, \xi_1; \vec{r}_2, \xi_2; \ldots; \vec{r}_i, \xi_i; \vec{r}_j, \xi_j; \ldots) =$$

position / spin variables of particle 1

$$= \Psi(\vec{r}_1, \xi_1; \vec{r}_2, \xi_2; \ldots; \vec{r}_j, \xi_j, \vec{r}_i, \xi_i, \ldots)$$

$$= \lambda\,\Psi(\vec{r}_1, \xi_1; \vec{r}_2, \xi_2; \ldots; \vec{r}_i, \xi_i; \vec{r}_j, \xi_j, \ldots)$$

constant to be determined

permute $i, j$

$$\hat{P}_{ij}\,(\hat{P}_{ij}\,\Psi) = \Psi = \lambda^2\Psi \Rightarrow \lambda^2 = 1 \Rightarrow \boxed{\lambda = \pm 1}$$

change system back to original state

The eigenvectors of $\hat{P}_{ij}$ with $\lambda = 1$ (5)

are called symmetric $\Rightarrow$ $\hat{P}_{ij} |\Psi_S\rangle = |\Psi_S\rangle$

Same with $\lambda = -1 \Rightarrow$ symmetric

$\hat{P}_{ij} |\Psi_A\rangle = -|\Psi_A\rangle$

antisymmetric $\rightarrow$ with respect to the exchange of two particles

<u>Symmetrization postulate</u>:

a system of identical particles is required to have a quantum state vector that is either symmetric or antisymmetric wrt exchange of any pair of particles.

The particles described by an antisymmetric state $\Psi_A$ $\Rightarrow$ fermions ; symmetric $\Psi_S$ $\Rightarrow$ bosons

The physical criterion that distinguishes between the two kinds of particles is their spin:

fermions have half-integer spin

bosons have integer spin

<u>Ex. of fermions</u>: $e^-$, proton, neutron, neutrino, $^{13}C$ nucleus
(all spin-1/2)

Ex. of bosons: photons (spin-1), $\pi$-meson (spin 0), deuterons (spin 1), $\alpha$-particles (spin-0),

deuterons: 1p+1n; $\alpha$-particles: 2p+2n; Oxygen nuclei (spin 0)

Note: • bosons can't become fermions & vice versa with any interaction

• Generally, systems with even (odd) # of fermions will behave as bosons (fermions)

The antisymmetry of the fermion wave function is equivalent to the Pauli's exclusion principle $\Rightarrow$ in a system of electrons (or any fermions), two electrons can never simultaneously occupy the same state $\psi_{n \ell m_\ell m_s}$.

Back to our Ex. from p.3 $\Rightarrow |ab\rangle, |ba\rangle$, or combination?

$|\psi_A\rangle = (|ab\rangle - |ba\rangle)\frac{1}{\sqrt{2}}$ if our particles are fermions

Check: $P_{12}|\psi_A\rangle = (|ba\rangle - |ab\rangle)\frac{1}{\sqrt{2}} = -(|ab\rangle - |ba\rangle)\frac{1}{\sqrt{2}}$

$|\psi_S\rangle = (|ab\rangle + |ba\rangle)\frac{1}{\sqrt{2}}$ if our particles are bosons

Check: $P_{12}|\psi_S\rangle = |\psi_S\rangle$

More generally: $|\psi\rangle = |\psi_{spatial}\rangle\,|\psi_{spin}\rangle$

| | $\lvert\psi\rangle$ | $\lvert\psi_{spatial}\rangle$ | $\lvert\psi_{spin}\rangle$ |
|---|---|---|---|
| bosons | S | S | S |
| bosons | S | A | A |
| fermions | A | A | S |
| fermions | A | S | A |