RANDOM CASCADES:
A Stochastic Model
Meets Some Data

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INFORMATICS:

extracting conceptual information from data

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Today's talk: A stochastic model

What is its correspondence with data?

CAN ONE INFORM THE OTHER?
Definition of **MODEL**:

Equation with variables representing physical quantities

Stochastic model incorporates random variability

Look at ‘random cascade’ model in 2 physical settings:

- *Energy dissipation in turbulence*

- *Precipitation intensity over time*
IDEA MOTIVATING MODEL:

total energy splits randomly repeats independently • • • until inertial range
A more mathematical sketch...

- Total energy
- Splits randomly
- Repeats independently
ORIGINATION OF IDEA: Kolmogorov in the early 1940’s

PHYSICAL SETTING: energy dissipation in turbulence

Russian school: 1940’s -1960’s ... turbulent fluids

French mathematicians: 1970’s ... mathematical rigor

Since then: appeared in different physical contexts

- precipitation intensity

- internet traffic intensity
BUILDING MODEL: Divide region into pixels successively

\[ \Delta_n: \text{nth stage pixel, area } b^{-n}, \text{ } b \text{ integer reflecting dimension} \]

\[ R_n(\Delta_n) = \frac{1}{b} \sum_{i=0}^{b-1} W_i(\Delta_n) R_{n+1}(\Delta_{n+1}) \]

\( W \)'s are random "splitting" weights

\( R_n \)'s "fine scale" unobservable random structure

Assumptions:

- this structure

- \( W \)'s independent with same variability across region and scale
Observable data: Random measure of pixels

\[ R_0(\Delta_n) = \prod_{k=0}^{n-1} W(\Delta_k) \frac{1}{b} \sum_{i=0}^{b-1} W_i(\Delta_n) R_{n+1}(\Delta_{n+1}) \]

Properties

- intermittency
- stochastic self-similarity
- multi-fractal structure
Model produces a random measure for region ... who is interested and why?

Mathematicians: What are the distributional and geometric properties of model?

Physicists: modeling energy dissipation in turbulence: What is the distribution of the random splitting mechanism?

Hydrologists: Can this model be used to simulate realistic precipitation fields?

Electrical engineers: Does a model of this nature reflect internet usage patterns?

Mathematical statisticians: What estimation and testing procedures can be applied to data to help inform the questions above?
Start with the physicist’s question:

Rephrased, it becomes ... Can the distribution of the $W$’s be reclaimed from data on the random measure over a given region?

$$R_n(\Delta_n) = \frac{1}{b} \sum_{i=0}^{b-1} W_i(\Delta_n) R_{n+1}(\Delta_{n+1})$$

**OBSERVABLE DATA:**

$$R_0(\Delta_n) = \prod_{k=0}^{n-1} W(\Delta_k) \frac{1}{b} \sum_{i=0}^{b-1} W_i(\Delta_n) R_{n+1}(\Delta_{n+1})$$
KEY MATH TOOL: knowledge of $E(W^h)$ for all $h$ close to 0 determines the distribution of $W$

Get at this by looking at:

$$M_n(h) = \sum_{\Delta_n} R_0^h(\Delta_n)$$ empirical moments

Log transform easier to work with: $$\hat{\chi}_n(h) = \frac{\log_b M_n(h)}{n}$$

Gives estimates of the STRUCTURE FUNCTION:

$$\chi(h) = \log_b(EW^h) - (h - 1)$$
Kolmogorov: the W’s in turbulence are log-normal
CORRESPONDS TO A QUADRATIC $\chi$
STRUCTURE FUNCTION
Does observed data follow this pattern?

data from Anselmet et al, 1984
She and Levesque give an alternate hypothesis in 1994; results in log-Poisson $W$'s (and a different structure function)
Mathematicians weigh in: \( \hat{\chi}_n(h) \) DOESN’T ESTIMATE \( \chi(h) \) FOR ALL \( h \)!
So, are the \( W \)'s log-normal or not?

**Problem:** available data is all in the form of the average empirical moments \( M_n(h) \)'s

Need estimates of the variability of the \( M_n(h) \)'s

Mathematicians have the necessary central limit theorem ready and waiting!
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Precipitation question .... simulating rainfall data.
First step: time-warp

Observed and normalized Corvallis daily rainfall, 1995-99.
Typical diagnostic: \( \hat{\chi} \) versus log of \( \Delta_n \) area
In another setting:
Another diagnostic: look at ratios $R_0(\Delta_{n+1})/R_0(\Delta_n)$

Partition coefficients for temporal rainfall intensity
15 minutes (upper left) to 64 hours (lower right)
Alternate model

Distribution of random weights depends on scale

\[ R_n(\Delta_n) = \frac{1}{b} \sum_{i=0}^{b-1} W_{n,i}(\Delta_n) R_{n+1}(\Delta_{n+1}) \]

Note: this is a harder estimation problem.

Question (for the geophysicists): Is there an underlying physical argument that can justify this?
Idea: spatial random cascade moving over landscape
A simulation:
Model

⇓

Data Analysis

⇓

Re-examine model ⇒

⇔ New Model

⇑

Inform understanding of underlying phenomena

⇑

Re-examine data