MTH 312: THEOREMS AND DEFINITIONS

The lower and upper Darboux sums are respectively

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \qquad \text{and} \qquad U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

where $P = \{x_0, ..., x_n\}$ is a partition of $[a,b]$,

$$m_i = \inf_{x_{i-1} \le x \le x_i} f(x) \qquad \text{and} \qquad M_i = \sup_{x_{i-1} \le x \le x_i} f(x).$$

The Refinement Lemma: For any partition P and refinement P^*

$$L(f,P) \le L(f,P^*) \le U(f,P^*) \le U(f,P).$$

$$\underline{\int_a^b} f = \sup_P L(f,P) \qquad \overline{\int_a^b} f = \inf_P U(f,P)$$

Definition: $f : [a,b] \to \mathbf{R}$ is **integrable** if $\underline{\int_a^b} f = \overline{\int_a^b} f$.

The Archimedes-Riemann Theorem: A bounded $f : [a,b] \to \mathbf{R}$ is integrable if and only if there is a sequence of partitions P_n of $[a,b]$ with

$$\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0.$$

In that case

$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n) = \int_a^b f.$$

The Darboux Sum Convergence Theorem: If $f : [a,b] \to \mathbf{R}$ is integrable, any sequence of partitions P_n of $[a,b]$ with $\lim_{n\to\infty} \text{gap } P_n = 0$ has

$$\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0.$$

The Riemann Sum Convergence Theorem: If $f : [a,b] \to \mathbf{R}$ is integrable, any sequence of partitions P_n of $[a,b]$ with $\lim_{n\to\infty} \text{gap } P_n = 0$ has

$$\lim_{n\to\infty} R(f,P_n,C_n) = \int_a^b f.$$

The First Fundamental Theorem of Calculus: If $F : [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with $F' : (a,b) \to \mathbf{R}$ both continuous and bounded, then

$$F(b) - F(a) = \int_a^b F'.$$

The Second Fundamental Theorem of Calculus: If $f : [a, b] \to \mathbf{R}$ is continuous then

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x) \qquad \text{for } x \in (a, b).$$

The Mean Value Theorem for Integrals: If $f : [a, b] \to \mathbf{R}$ is continuous, then there is some $c \in [a, b]$ with

$$\frac{1}{b-a}\int_a^b f = f(c).$$

The Triangle Inequality for Integrals: If $f : [a, b] \to \mathbf{R}$ is integrable, then

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

Definition: Functions f and g have **contact of order** n at x_0 if they are both n times differentiable at x_0 with

$$f^{(k)}(x_0) = g^{(k)}(x_0) \qquad \text{for} \quad k = 0, 1, ..., n.$$

Definition: The **Taylor polynomial of degree** n **for** f **at** x_0 is

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

The Lagrange Remainder Theorem: For some c_n between x_0 and x,

$$f(x) = p_n(x) + \frac{f^{(n+1)}(c_n)}{(n+1)!}(x - x_0)^{n+1}.$$

The Cauchy Remainder Theorem:

$$f(x) = p_n(x) + \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)dt.$$

Lemma: For any c,

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0.$$

Theorem: If I is an open interval with $[x_0 - r, x_0 + r] \subset I$ and the function f has derivatives of all orders on I with $|f^{(n)}(x)| \leq M^n$ for all $x \in [x_0 - r, x_0 + r]$, $n \geq 0$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

for all $x \in [x_0 - r, x_0 + r]$.