A Brief Introduction to Hilbert Spaces

A **Hilbert space** H is a vector space with field F and **inner product** $< \cdot, \cdot >: H \to F$ that is complete with respect to the norm induced by the inner product, $||x|| = \sqrt{< x, x >}$.

The inner product satisfies the **Cauchy-Schwarz inequality**:

$$| < x, y > | \leq ||x|| \, ||y||.$$

A subspace $M \subset H$ is a subset of H that is itself a vector space. $E \subset H$ is convex if it is closed under the formation of convex combinations. Subspaces are convex; subspaces shifted by a fixed element of H are also convex.

Proposition: If $E \subset H$ is convex, closed, and non-empty, then E contains a unique element x_E with

$$||x_E|| = \inf_{x \in E} ||x||.$$

If $< x, y >= 0$, we say x is **orthogonal** to y and write $x \perp y$. For $x \in H$ we define the set

$$x^\perp = \{y \in H : y \perp x\}$$

and for a set $M \subset H$,

$$M^\perp = \cap_{x \in M} x^\perp.$$

Note: If M is a closed subspace, then any $x \in H$ can be decomposed uniquely

$$x = P_M x + P_{M^\perp} x$$

where $P_M x \in M$ and $P_{M^\perp} x \in M^\perp$. $P_M x$ is often referred to as the projection of x into the subspace M.

The (other) Riesz Representation Theorem: If $L : H \to F$ is a bounded linear functional, then there is a unique $y = y_L \in H$ with

$$Lx =< x, y > \quad \text{for all} \quad x \in H.$$

Proof: Since H is a Banach space and L is a bounded linear functional, L is continuous. If $L^{-1}(\{0\}) = H$, we can take $y = 0$. Otherwise, let $M = L^{-1}(\{0\})$. It is easy to see that M is a closed and proper subspace of H. Take an arbitrary nonzero $z \in M^\perp$ and let

$$y = \frac{\overline{Lz}}{||z||^2} \, z.$$

Notice that $y \in M^\perp$ with

$$Ly = \frac{\overline{Lz}}{||z||^2} \, Lz = \frac{|Lz|^2}{||z||^2} = ||y||^2 =< y, y > .$$

Now take an arbitrary $x \in H$ and let

$$w = w(x) = x - \frac{Lx}{||y||^2} y$$

and notice that

$$Lw = Lx - \frac{Lx}{||y||^2} Ly = Lx - Lx = 0$$

That is, $w \in M$ so $w \perp y$ and

$$< x, y > = < x - w, y > = < \frac{Lx}{||y||^2} y, y > = Lx. \qquad \square$$

A collection $\{u_\alpha : \alpha \in A\}$ is **orthonormal** if $u_\alpha \perp u_\beta$ for $\alpha \neq \beta$ and $||u_\alpha|| = 1$ for all $\alpha \in A$. If H is contained in the linear span of $\{u_\alpha : \alpha \in A\}$, then we say it is an **orthonormal basis** for H. (Every Hilbert space has an orthonormal basis.)

Bessel's inequality tells us that for any orthonormal $\{u_\alpha : \alpha \in A\}$

$$\Sigma_{\alpha \in A} | < x, u_\alpha > |^2 \leq ||x||^2.$$

Parseval's identity tells us that, if $\{u_\alpha : \alpha \in A\}$ is an orthonormal basis for H, then

$$\Sigma_{\alpha \in A} | < x, u_\alpha > |^2 = ||x||^2.$$

Three problems adapted from Bass.

H.1 (Bass 19.2) Let H be a Hilbert space and $\{x_n\} \subset H$ be a sequence satisfying $||x_n|| \to ||x||$ and $< x_n, y > \to < x, y >$ for all $y \in H$ as $n \to \infty$. Show that $x_n \to x$ as $n \to \infty$.

H.2 (Bass 19.6) Let H be a Hilbert space with a countably infinite orthonormal basis $\{u_k : k \geq 1\}$. Show that the closed unit ball in H is not compact.

H.3 (Bass 19.13) Let H be a Hilbert space with a countably infinite orthonormal basis $\{u_k : k \geq 1\}$ and let $\{v_k : k \geq 1\}$ be an orthonormal subset of H with

$$\Sigma_{k \geq 1} ||u_k - v_k||^2 < 1.$$

Show that $\{v_k : k \geq 1\}$ must also be an orthonormal basis for H.