#15. Def: \( c = \max S \) iff \( c \in S \) and \( c \) is an upper bound for \( S \).

Lemma: \( S \) has a maximum iff it is bounded above and \( \text{sup} S \in S \).

Proof: \( \Rightarrow \) Let \( S \) have a maximum \( c \). Then from def. \( c \) is an upper bound \( \Rightarrow \) \( S \) is bounded from above.
To show \( \text{sup} S \in S \) we show \( c = \text{sup} S \). We must have \( x \leq c, \forall x \in S \), hence \( \text{sup} S \leq c \) (\( \text{sup} S \) is the l.u.b.).
Since \( c \in S \), we have \( c \leq \text{sup} S \) (\( \text{sup} S \) is an upper bound).
From \( \text{sup} S \leq c \) and \( c \leq \text{sup} S \) we conclude \( c = \text{sup} S \).

\( \Leftarrow \) Let \( S \) be bounded from above, and \( \text{sup} S \in S \).
Set \( c = \text{sup} S \in S \). This \( c \) satisfies the def above hence \( c = \max S \), and \( S \) has a maximum.

q.e.d.