16.1.1 The potential for the harmonic oscillator is

$$V(x)=\tfrac{1}{2}m\omega^2x^2$$

Let's try a Gaussian trial wave function in the variational method to estimate the ground state energy and ground state wave function. The Hamiltonian is

$$H=T+V(x)=\frac{p^2}{2m}+\tfrac{1}{2}m\omega^2x^2$$

The trial function is

$$\psi(x,\alpha)=e^{-\alpha x^2}$$

The energy functional

$$E[\psi]=\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}\geq E_0$$

is then

$$E[\psi]=\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}=\frac{\int e^{-\alpha x^2}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\tfrac{1}{2}m\omega^2x^2\right)e^{-\alpha x^2}\,dx}{\int e^{-2\alpha x^2}\,dx}$$

The denominator is

$$\int_{-\infty}^{\infty}e^{-2\alpha x^2}\,dx=\sqrt{\frac{\pi}{2\alpha}}$$

The numerator is

$$\begin{aligned}\int_{-\infty}^{\infty}e^{-\alpha x^2}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\tfrac{1}{2}m\omega^2x^2\right)e^{-\alpha x^2}\,dx&=\int_{-\infty}^{\infty}e^{-\alpha x^2}\left(-\frac{\hbar^2}{2m}\left(4\alpha^2x^2-2\alpha\right)+\tfrac{1}{2}m\omega^2x^2\right)e^{-\alpha x^2}\,dx\\&=\int_{-\infty}^{\infty}\left(\frac{\hbar^2\alpha}{m}+\left(-\frac{2\hbar^2\alpha^2}{m}+\tfrac{1}{2}m\omega^2\right)x^2\right)e^{-2\alpha x^2}\,dx\\&=\frac{\hbar^2\alpha}{m}\sqrt{\frac{\pi}{2\alpha}}+\left(-\frac{2\hbar^2\alpha^2}{m}+\tfrac{1}{2}m\omega^2\right)\frac{1}{4}\sqrt{\frac{\pi}{2\alpha^3}}\\&=\sqrt{\frac{\pi}{2\alpha}}\left(\frac{\hbar^2\alpha}{2m}+\frac{m\omega^2}{8\alpha}\right)\end{aligned}$$

Hence, the energy function is

$$E(\alpha)=\frac{\hbar^2\alpha}{2m}+\frac{m\omega^2}{8\alpha}$$

Minimize:

$$\frac{d}{d\alpha}E(\alpha)=\frac{\hbar^2}{2m}-\frac{m\omega^2}{8\alpha^2}=0\ \Rightarrow\ \alpha_0=\frac{m\omega}{2\hbar}$$

The minimum energy is thus

$$E(\alpha_0)=\frac{\hbar\omega}{4}+\frac{\hbar\omega}{4}=\frac{1}{2}\hbar\omega$$

which turns out to the exact (known) answer.

16.1.3 The potential for the delta function system is

$$V(x)=-aV_0\delta(x)$$

Let's try a Gaussian trial wave function in the variational method to estimate the ground state energy. The Hamiltonian is

$$H=T+V(x)=\frac{p^2}{2m}-aV_0\delta(x)$$

The trial function is

$$\psi(x,\alpha)=e^{-\alpha x^2}$$

The energy functional

$$E[\psi]=\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}\geq E_0$$

is then

$$E[\psi]=\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}=\frac{\int e^{-\alpha x^2}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}-aV_0\delta(x)\right)e^{-\alpha x^2}\,dx}{\int e^{-2\alpha x^2}\,dx}$$

The denominator is

$$\int_{-\infty}^{\infty}e^{-2\alpha x^2}\,dx=\sqrt{\frac{\pi}{2\alpha}}$$

The numerator is

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}-aV_0\delta(x)\right)e^{-\alpha x^2}\,dx=\int_{-\infty}^{\infty} e^{-\alpha x^2}\left(-\frac{\hbar^2}{2m}\left(4\alpha^2x^2-2\alpha\right)-aV_0\delta(x)\right)e^{-\alpha x^2}\,dx$$

$$=\int_{-\infty}^{\infty}\left(\frac{\hbar^2\alpha}{m}-\frac{2\hbar^2\alpha^2}{m}x^2-aV_0\delta(x)\right)e^{-2\alpha x^2}\,dx$$

$$=\frac{\hbar^2\alpha}{m}\sqrt{\frac{\pi}{2\alpha}}+\left(-\frac{2\hbar^2\alpha^2}{m}\right)\frac{1}{4}\sqrt{\frac{\pi}{2\alpha^3}}-aV_0$$

$$=\sqrt{\frac{\pi}{2\alpha}}\left(\frac{\hbar^2\alpha}{2m}\right)-aV_0$$

Hence, the energy function is

$$E(\alpha)=\frac{\hbar^2\alpha}{2m}-aV_0\sqrt{\frac{2\alpha}{\pi}}$$

Minimize:

$$\frac{d}{d\alpha}E(\alpha)=\frac{\hbar^2}{2m}-aV_0\frac{1}{2}\sqrt{\frac{2}{\pi\alpha}}=0\;\Rightarrow\;\alpha_0=\frac{2m^2a^2V_0^2}{\pi\hbar^4}$$

The minimum energy is thus

$$E(\alpha_0)=\frac{\hbar^2\alpha}{2m}-aV_0\sqrt{\frac{2\alpha}{\pi}}=\frac{\hbar^2}{2m}\frac{2m^2a^2V_0^2}{\pi\hbar^4}-aV_0\sqrt{\frac{2}{\pi}}\frac{\sqrt{2}maV_0}{\sqrt{\pi}\hbar^2}$$

$$=\frac{ma^2V_0^2}{\pi\hbar^2}-\frac{2ma^2V_0^2}{\pi\hbar^2}=-\frac{ma^2V_0^2}{\pi\hbar^2}\simeq-(0.32)\frac{ma^2V_0^2}{\hbar^2}$$

The exact (known) answer is

$$E_{gnd}=-\frac{ma^2V_0^2}{2\hbar^2}=-(0.5)\frac{ma^2V_0^2}{\hbar^2}$$

so our estimate is somewhat higher, as expected.

16.2.4 The wave function on the outside of the barrier (at x_e) in terms of the value inside (at x_0) is (Eq.16.2.24)

$$\psi(x_e)=\psi(x_0)\exp\left[\frac{i}{\hbar}\int_{x_0}^{x_e} i\sqrt{2m\left(V(x)-E\right)}\,dx\right]\equiv\psi(x_0)e^{-\gamma/2}$$

The lifetime is (Eq. 16.2.26, with $V_0=0$ for our case)

$$\tau=\frac{1}{R}=\frac{2mx_0}{\sqrt{2mE}}e^{\gamma}$$

The potential is just the Coulomb potential of the alpha particle in the presence of the nucleus with charge Z after emission:

$$V(x)=\frac{(Z_\alpha e)(Ze)}{x}=\frac{2Ze^2}{x}$$

For an alpha particle of energy E, the outer turning point is

$$\frac{2Ze^2}{x_e}=E \Rightarrow x_e=\frac{2Ze^2}{E}$$

Now find γ :

$$\gamma=-2\frac{i}{\hbar}\int_{x_0}^{x_e} i\sqrt{2m(V(x)-E)}\,dx=\frac{2}{\hbar}\int_{x_0}^{x_e}\sqrt{2m\left(\frac{2Ze^2}{x}-E\right)}\,dx$$
$$=\frac{2}{\hbar}\int_{x_0}^{x_e}\sqrt{2m\left(\frac{x_eE}{x}-E\right)}\,dx=\frac{2}{\hbar}\sqrt{2mE}\int_{x_0}^{x_e}\sqrt{\left(\frac{x_e}{x}-1\right)}\,dx$$

Let $u=x/x_e$ and $y=x_0/x_e$, and note that $v=\sqrt{2E/m}$:

$$\gamma=\frac{2x_e}{\hbar}\sqrt{2mE}\int_y^1\sqrt{\left(\frac{1}{u}-1\right)}\,du=\frac{2}{\hbar}\frac{2Ze^2}{E}\sqrt{2mE}\int_y^1\sqrt{\left(\frac{1}{u}-1\right)}\,du$$
$$=\frac{8Ze^2}{\hbar v}\int_y^1\sqrt{\left(\frac{1}{u}-1\right)}\,du$$

To do the integral, make the substitution $u=\sin^2 z$

$$\gamma=\frac{8Ze^2}{\hbar v}\int_{\sin^{-1}\sqrt{y}}^{\pi/2}\sqrt{\left(\frac{1}{\sin^2 z}-1\right)}2\sin z\cos z\,dz=\frac{8Ze^2}{\hbar v}2\int_{\sin^{-1}\sqrt{y}}^{\pi/2}\cos^2 z\,dz$$
$$=\frac{8Ze^2}{\hbar v}2\left[\frac{z}{2}+\frac{1}{4}\sin 2z\right]_{\sin^{-1}\sqrt{y}}^{\pi/2}=\frac{8Ze^2}{\hbar v}\left[z+\sin z\cos z\right]_{\sin^{-1}\sqrt{y}}^{\pi/2}$$
$$=\frac{8Ze^2}{\hbar v}\left[\frac{\pi}{2}-\sin^{-1}\sqrt{y}-\sin\left(\sin^{-1}\sqrt{y}\right)\cos\left(\sin^{-1}\sqrt{y}\right)\right]$$
$$=\frac{8Ze^2}{\hbar v}\left[\frac{\pi}{2}-\sin^{-1}\sqrt{y}-\sqrt{y}\sqrt{1-y}\right]$$
$$=\frac{8Ze^2}{\hbar v}\left[\cos^{-1}\sqrt{y}-\sqrt{y}\sqrt{1-y}\right]$$

Let's first estimate y. It is convenient to recall that $13.6eV=e^2/2a_0$:

$$y=\frac{x_0}{x_e}=\frac{x_0}{\frac{2Ze^2}{E}}=\frac{Ex_0}{2Ze^2}=\frac{4.2MeV\left(10^{-12}cm\right)}{2(90)(2\times0.0529nm)13.6eV}=0.162$$

Hence we can use the small angle approximation on the trig function to get

$$\gamma \simeq \frac{8Ze^2}{\hbar v}\left[\frac{\pi}{2}-\sqrt{y}-\sqrt{y(1-y)}\right] \simeq \frac{8Ze^2}{\hbar\sqrt{2E/m}}\left[\frac{\pi}{2}-\sqrt{y}-\sqrt{y(1-y)}\right]$$
$$\simeq \frac{8Z}{\sqrt{2}}\frac{e^2}{\hbar c}\sqrt{\frac{mc^2}{E}}\left[\frac{\pi}{2}-\sqrt{y}-\sqrt{y(1-y)}\right] \simeq \frac{8Z}{\sqrt{2}}\alpha\sqrt{\frac{mc^2}{E}}\left[\frac{\pi}{2}-\sqrt{y}-\sqrt{y(1-y)}\right]$$

Putting in numbers gives (m is the mass of the alpha: 3.73 GeV/c^2)

$$\gamma \simeq \frac{8(90)}{\sqrt{2}}\frac{1}{137}\sqrt{\frac{3.73GeV}{4.2MeV}}\left[\frac{\pi}{2}-\sqrt{0.162}-\sqrt{0.162(1-0.162)}\right] \simeq 110.7[0.7998]$$
$$\simeq 88.5$$

and a lifetime

$$\tau = \frac{1}{R} = \frac{2mx_0}{\sqrt{2mE}}e^{\gamma} = \sqrt{2}\frac{\sqrt{mc^2}}{\sqrt{E}}\frac{x_0}{c}e^{\gamma}$$
$$\simeq \sqrt{2}\sqrt{\frac{3.73GeV}{4.2MeV}}\frac{10^{-12}cm}{3\times10^{10}cms^{-1}}e^{88.5} \simeq 3.8\times10^{17}s \simeq 1.2\times10^{10}\,years$$

16.2.7 The bound states in the WKB approximation are found by integrating the momentum between the classical turning points:

$$\int_{x_1}^{x_2} p(x)dx = \left(n+\tfrac{1}{2}\right)\pi\hbar$$

For the harmonic oscillator the momentum is

$$E = \frac{p^2}{2m}+\tfrac{1}{2}m\omega^2x^2 \;\Rightarrow\; p(x) = \sqrt{2m\left(E-\tfrac{1}{2}m\omega^2x^2\right)} = m\omega\sqrt{\frac{2E}{m\omega^2}-x^2}$$

and the turning points are

$$p(x_{1,2}) = 0 = m\omega\sqrt{\frac{2E}{m\omega^2}-x^2} \;\Rightarrow x_2 = -x_1 = \sqrt{\frac{2E}{m\omega^2}}$$

Applying the WKB condition gives

$$\left(n+\tfrac{1}{2}\right)\pi\hbar = \int_{x_1}^{x_2} p(x)dx = m\omega\int_{-x_2}^{x_2}\sqrt{x_2^2-x^2}\,dx$$
$$= m\omega\frac{1}{2}\left[x\sqrt{x_2^2-x^2}+x_2^2\sin^{-1}\left(\frac{x}{x_2}\right)\right]_{-x_2}^{x_2} = m\omega\frac{1}{2}\left[x_2^2\frac{\pi}{2}-x_2^2\left(-\frac{\pi}{2}\right)\right]$$
$$= \frac{1}{2}m\omega\pi x_2^2 = \frac{1}{2}m\omega\pi\frac{2E}{m\omega^2} = \pi\frac{E}{\omega}$$

Solving for the energy gives

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega$$

which is the exact answer.