1. a,b) Time-dependent perturbation theory gives us the probability amplitude for the final state:

$$c_{n_f}(t) = \frac{1}{i\hbar}\int_0^t \langle n_f | \hat{H}'(t') | n_i \rangle e^{i(E_f - E_i)t'/\hbar}\, dt' = \frac{1}{i\hbar}\int_0^t \langle n | Ax^3 e^{-\gamma t'} | 0 \rangle e^{i(E_f - E_i)t'/\hbar}\, dt'$$

$$= \frac{1}{i\hbar}\langle n | Ax^3 | 0 \rangle \int_0^t e^{-\gamma t'} e^{i(E_n - E_0)t'/\hbar}\, dt' = \frac{A}{i\hbar} x_{n0}^3 \int_0^t e^{-\gamma t'} e^{i(E_n - E_0)t'/\hbar}\, dt'$$

To find the matrix elements use ladder operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right)$$

$$x^3 = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(a^\dagger + a\right)^3$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(a^\dagger a^\dagger a^\dagger + a^\dagger a^\dagger a + a^\dagger a a^\dagger + a^\dagger a a + a a^\dagger a^\dagger + a a^\dagger a + a a a^\dagger + a a a\right)$$

When we take the matrix elements, half of these will vanish because $a|0\rangle = 0$, so any term with an a rightmost vanishes. The $aaa^\dagger$ term is also zero since it tries to lower the ground state on the last ladder operation. Thus we are left with

$$\langle n | x^3 | 0 \rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \langle n | \left(a^\dagger a^\dagger a^\dagger + a^\dagger a a^\dagger + a a^\dagger a^\dagger\right) | 0 \rangle$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\sqrt{1}\sqrt{2}\sqrt{3}\delta_{n3} + \sqrt{1}\sqrt{1}\sqrt{1}\delta_{n1} + \sqrt{1}\sqrt{2}\sqrt{2}\delta_{n1}\right) = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\sqrt{6}\delta_{n3} + 3\delta_{n1}\right)$$

Now do the time integral, noting that the harmonic oscillator energies are $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$

$$c_n(t) = \frac{A}{i\hbar} x_{n0}^3 \int_0^t e^{-\gamma t'} e^{i(E_n - E_0)t'/\hbar}\, dt' = \frac{A}{i\hbar} x_{n0}^3 \int_0^t e^{-\gamma t'} e^{in\omega t'}\, dt' = \frac{A}{i\hbar} x_{n0}^3 \int_0^t e^{in\omega t' - \gamma t'}\, dt'$$

$$= \frac{A}{i\hbar} x_{n0}^3 \left[\frac{e^{in\omega t' - \gamma t'}}{in\omega - \gamma}\right]_0^t = \frac{A}{i\hbar} x_{n0}^3 \left[\frac{e^{in\omega t - \gamma t} - 1}{in\omega - \gamma}\right]$$

For times longer than 1/γ, the exponential term goes to zero, giving

$$c_n\left(t >> \tfrac{1}{\gamma}\right) = \frac{A}{i\hbar} x_{n0}^3 \left[\frac{-1}{in\omega - \gamma}\right]$$

$$\mathcal{P}_n(t) = |c_n(t)|^2 = \frac{A^2}{\hbar^2}\left|x_{n0}^3\right|^2 \left[\frac{1}{in\omega - \gamma} \cdot \frac{1}{-in\omega - \gamma}\right]$$

$$= \frac{A^2}{\hbar^2}\frac{1}{n^2\omega^2 + \gamma^2}\left(\frac{\hbar}{2m\omega}\right)^3\left(\sqrt{6}\delta_{n3} + 3\delta_{n1}\right)^2 = \frac{3A^2\hbar}{8m^3\omega^3}\frac{1}{n^2\omega^2 + \gamma^2}\left[2\delta_{n3} + 3\delta_{n1}\right]$$

The Kronecker delta functions tells us the selection rules:

$$\boxed{\delta_{n3}, \delta_{n1} \Rightarrow n = 1, 3}$$

2. a) For a hard sphere the wave function inside must be 0. Matching this to the wave function outside the sphere gives

$$\cos\delta_\ell \; j_\ell(kr_0) - \sin\delta_\ell \; \eta_\ell(kr_0) = 0$$

Solving for the phase gives

$$\tan\delta_\ell = \frac{j_\ell(kr_0)}{\eta_\ell(kr_0)}$$

$$\delta_\ell = \tan^{-1}\left[\frac{j_\ell(kr_0)}{\eta_\ell(kr_0)}\right]$$

b) For *s*-wave scattering, the phase shift is

$$\delta_0 = \tan^{-1}\left[\frac{j_0(kr_0)}{\eta_0(kr_0)}\right] = \tan^{-1}\left[\frac{\frac{\sin kr_0}{kr_0}}{-\frac{\cos kr_0}{kr_0}}\right] = \tan^{-1}\left[-\frac{\sin kr_0}{\cos kr_0}\right] = \tan^{-1}\left[-\tan kr_0\right]$$

$$= -kr_0$$

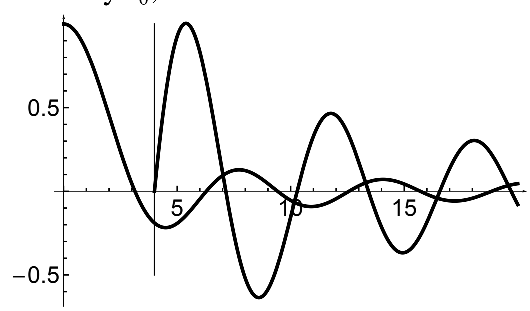
To understand, write the wave function with this phase shift

$$R(r) = A\left[\cos\delta_0 \; j_0(kr) - \sin\delta_0 \; \eta_0(kr)\right] = A\left[\cos\delta_0 \; \frac{\sin kr}{kr} + \sin\delta_0 \; \frac{\cos kr}{kr}\right]$$

$$= \frac{A}{kr}\left(\sin kr\cos\delta_0 + \cos kr\sin\delta_0\right) = \frac{A}{kr}\sin\left(kr + \delta_0\right)$$

$$= \frac{A}{kr}\sin\left(kr - kr_0\right) = \frac{A}{kr}\sin k\left(r - r_0\right)$$

Wave function is pushed out by r_0, as shown below



For small energy ($kr_0 \ll 1$), *s*-wave scattering dominates and the total cross-section becomes

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell \cong \frac{4\pi}{k^2}\sin^2\delta_0$$

$$\cong \frac{4\pi}{k^2}\sin^2 kr_0 \cong \frac{4\pi}{k^2}\left(kr_0\right)^2$$

$$\cong 4\pi r_0^2$$

This is 4 times larger than the classical cross-section of πr_0^2

c) For p-wave scattering, the phase shift is

$$\delta_1 = \tan^{-1}\left[\frac{j_1(kr_0)}{\eta_1(kr_0)}\right] = \tan^{-1}\left[\frac{\frac{\sin kr_0}{(kr_0)^2} - \frac{\cos kr_0}{kr_0}}{-\frac{\cos kr_0}{(kr_0)^2} - \frac{\sin kr_0}{kr_0}}\right]$$

$$= \tan^{-1}\left[\frac{kr_0 \cos kr_0 - \sin kr_0}{kr_0 \sin kr_0 + \cos kr_0}\right]$$

For small energy ($kr_0 \ll 1$), expand the trig functions

$$\delta_1 = \tan^{-1}\left[\frac{kr_0 \cos kr_0 - \sin kr_0}{kr_0 \sin kr_0 + \cos kr_0}\right] \cong \tan^{-1}\left[\frac{kr_0\left[1-(kr_0)^2/2\right]-\left[kr_0-(kr_0)^3/6\right]}{kr_0\left[kr_0-(kr_0)^3/6\right]+\left[1-(kr_0)^2/2\right]}\right]$$

$$\cong \tan^{-1}\left[\frac{-(kr_0)^3/3}{1+(kr_0)^2/2}\right] \cong -(kr_0)^3/3$$

For small energy ($kr_0 \ll 1$), this is much smaller than the s-wave phase shift and so can be neglected.

3.

$$H \doteq V_0\begin{pmatrix} 3 & \varepsilon & 0 & 0 \\ \varepsilon & 3 & 2\varepsilon & 0 \\ 0 & 2\varepsilon & 5 & \varepsilon \\ 0 & 0 & \varepsilon & 7 \end{pmatrix}$$

a) For the unperturbed case we have

$$H_0 \doteq V_0\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

with eigenvalues $E_1 = 3V_0, E_2 = 3V_0, E_3 = 5V_0$, $E_4 = 7V_0$ and eigenvectors

$$|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |4\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that $|1\rangle$ and $|2\rangle$ are degenerate and $|3\rangle$ and $|4\rangle$ are nondegenerate.

b) Let's first do nondegenerate perturbation theory for the $|3\rangle$ and $|4\rangle$ states. First we need to write the perturbation Hamiltonian:

$$H' \doteq V_0 \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 2\varepsilon & 0 \\ 0 & 2\varepsilon & 0 & \varepsilon \\ 0 & 0 & \varepsilon & 0 \end{pmatrix}$$

The first-order corrections are

$$\begin{aligned} E_n^{(1)} &= \left\langle n^{(0)} \middle| \hat{H}' \middle| n^{(0)} \right\rangle \\ E_3^{(1)} &= \left\langle 3^{(0)} \middle| \hat{H}' \middle| 3^{(0)} \right\rangle = 0 \\ E_4^{(1)} &= \left\langle 4^{(0)} \middle| \hat{H}' \middle| 4^{(0)} \right\rangle = 0 \end{aligned}$$

So we need to go to second order for these states

$$E_n^{(2)} = \sum_{k\neq n} \frac{\left|\left\langle n^{(0)} \middle| \hat{H}' \middle| k^{(0)} \right\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$E_3^{(2)} = \sum_{k\neq n} \frac{\left|\left\langle 3^{(0)} \middle| \hat{H}' \middle| k^{(0)} \right\rangle\right|^2}{E_3^{(0)} - E_k^{(0)}} = \frac{\left|\left\langle 3^{(0)} \middle| \hat{H}' \middle| 1^{(0)} \right\rangle\right|^2}{E_3^{(0)} - E_1^{(0)}} + \frac{\left|\left\langle 3^{(0)} \middle| \hat{H}' \middle| 2^{(0)} \right\rangle\right|^2}{E_3^{(0)} - E_2^{(0)}} + \frac{\left|\left\langle 3^{(0)} \middle| \hat{H}' \middle| 4^{(0)} \right\rangle\right|^2}{E_3^{(0)} - E_4^{(0)}}$$

$$= \frac{|0|^2}{5V_0 - 3V_0} + \frac{|2\varepsilon V_0|^2}{5V_0 - 3V_0} + \frac{|\varepsilon V_0|^2}{5V_0 - 7V_0} = \varepsilon^2 V_0 \left(\frac{4}{2} + \frac{1}{-2} \right) = \frac{3}{2}\varepsilon^2 V_0$$

$$E_4^{(2)} = \sum_{k\neq n} \frac{\left|\left\langle 4^{(0)} \middle| \hat{H}' \middle| k^{(0)} \right\rangle\right|^2}{E_4^{(0)} - E_k^{(0)}} = \frac{\left|\left\langle 4^{(0)} \middle| \hat{H}' \middle| 1^{(0)} \right\rangle\right|^2}{E_4^{(0)} - E_1^{(0)}} + \frac{\left|\left\langle 4^{(0)} \middle| \hat{H}' \middle| 2^{(0)} \right\rangle\right|^2}{E_4^{(0)} - E_2^{(0)}} + \frac{\left|\left\langle 4^{(0)} \middle| \hat{H}' \middle| 3^{(0)} \right\rangle\right|^2}{E_4^{(0)} - E_3^{(0)}}$$

$$= \frac{|0|^2}{7V_0 - 3V_0} + \frac{|0|^2}{7V_0 - 3V_0} + \frac{|\varepsilon V_0|^2}{7V_0 - 5V_0} = \frac{1}{2}\varepsilon^2 V_0$$

Thus the energies to second order are

$$\boxed{\begin{aligned} E_3 &= V_0 \left[5 + \frac{3}{2}\varepsilon^2 \right] \\ E_4 &= V_0 \left[7 + \frac{1}{2}\varepsilon^2 \right] \end{aligned}}$$

Now look at the perturbation of the degenerate $|1\rangle$ and $|2\rangle$ states. Here we need to diagonalize the perturbation Hamiltonian within that 2x2 space:

$$H' \doteq V_0 \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 2\varepsilon & 0 \\ 0 & 2\varepsilon & 0 & \varepsilon \\ 0 & 0 & \varepsilon & 0 \end{pmatrix} \;\Rightarrow\; H'_{1,2} \doteq V_0 \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

Diagonalize this

$$\begin{vmatrix} -\lambda & \varepsilon V_0 \\ \varepsilon V_0 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\varepsilon V_0)^2 = 0$$

$$\lambda = \pm \varepsilon V_0$$

The corrected energies are

$$\boxed{\begin{aligned} E_1 &= E_1^{(0)} + E_1^{(1)} = 3V_0 + \varepsilon V_0 \\ E_2 &= E_2^{(0)} + E_2^{(1)} = 3V_0 - \varepsilon V_0 \end{aligned}}$$

4. The potential for the delta function system is

$$V(x) = -aV_0\delta(x)$$

The Hamiltonian is

$$H = T + V(x) = \frac{p^2}{2m} - aV_0\delta(x)$$

The trial function is

$$\psi(x) = \begin{cases} A(b+x) & -b < x < 0 \\ A(b-x) & 0 < x < b \\ 0 & |x| > b \end{cases}$$

The energy functional

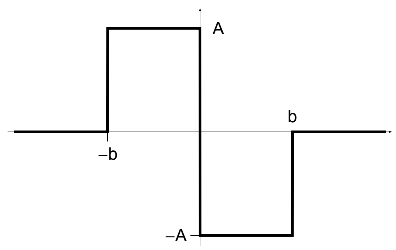
$$E[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

is then

$$E[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\int \psi^*(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - aV_0\delta(x)\right)\psi(x)\,dx}{\int |\psi(x)|^2\,dx}$$

The derivatives of the trial function require some care. They are

$$\frac{d}{dx}\psi(x) = \begin{cases} A & -b < x < 0 \\ -A & 0 < x < b \\ 0 & |x| > b \end{cases}$$



and

$$\frac{d^2}{dx^2}\psi(x) = A\delta(x+b) - 2A\delta(x) + A\delta(x-b)$$

The denominator of the energy functional is

$$\begin{aligned}\int_{-\infty}^{\infty}|\psi(x)|^2\,dx &= \int_{-b}^{0}|A(b+x)|^2\,dx + \int_0^b|A(b-x)|^2\,dx = 2\int_0^b|A(b-x)|^2\,dx \\ &= 2|A|^2\left[-\frac{1}{3}(b-x)^3\right]_0^b = 2|A|^2\left[0+\frac{b^3}{3}\right] \\ &= |A|^2\frac{2b^3}{3}\end{aligned}$$

The numerator is

$$\begin{aligned}\langle\psi|H|\psi\rangle &= \int\psi^*(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - aV_0\delta(x)\right)\psi(x)dx \\ &= \int_{-\infty}^{\infty}\psi^*(x)\left(-\frac{\hbar^2}{2m}\left(A\delta(x+b) - 2A\delta(x) + A\delta(x-b)\right) - aV_0\delta(x)\psi(x)\right)dx \\ &= -\frac{\hbar^2}{2m}\left[A\psi^*(-b) - 2A\psi^*(0) + A\psi^*(b)\right] - aV_0|\psi(0)|^2 \\ &= -\frac{\hbar^2}{2m}\left[-2AA^*b\right] - aV_0|A|^2b^2 \\ &= |A|^2\left(\frac{\hbar^2 b}{m} - aV_0b^2\right)\end{aligned}$$

Hence, the energy function is

$$\begin{aligned}E(b) &= \left(\frac{\hbar^2 b}{m} - aV_0b^2\right)\frac{3}{2b^3} \\ &= \frac{3\hbar^2}{2b^2m} - \frac{3aV_0}{2b}\end{aligned}$$

Minimize:

$$\frac{d}{db}E(b) = -\frac{3\hbar^2}{b^3m} + \frac{3aV_0}{2b^2} = 0 \;\Rightarrow\; b_0 = \frac{2\hbar^2}{maV_0}$$

The minimum energy is thus

$$E(b_0) = \frac{3\hbar^2}{2b_0^2 m} - \frac{3aV_0}{2b_0} = \frac{3\hbar^2}{2m}\frac{m^2a^2V_0^2}{4\hbar^4} - \frac{3aV_0}{2}\frac{maV_0}{2\hbar^2}$$

$$= \frac{3ma^2V_0^2}{8\hbar^2} - \frac{3ma^2V_0^2}{4\hbar^2} = -\frac{3ma^2V_0^2}{8\hbar^2} = -(0.375)\frac{ma^2V_0^2}{\hbar^2}$$

The exact (known) answer is

$$E_{gnd} = -\frac{ma^2V_0^2}{2\hbar^2} = -(0.5)\frac{ma^2V_0^2}{\hbar^2}$$

so our estimate is somewhat higher, as expected.