1. The expectation value of the energy is

$$\langle H\rangle = \frac{\langle P^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle X^2\rangle$$

Assume that $\langle P\rangle = 0$ and $\langle X\rangle = 0$ and use $\langle \Omega^2\rangle = (\Delta\Omega)^2 + \langle\Omega\rangle^2$ to get

$$\langle H\rangle = \frac{(\Delta P)^2}{2m} + \frac{1}{2}m\omega^2(\Delta X)^2$$

Assuming the uncertainty relation $\Delta P\Delta X \ge \hbar/2$, we then have

$$\langle H\rangle \ge \frac{\hbar^2}{8m(\Delta X)^2} + \frac{1}{2}m\omega^2(\Delta X)^2$$

Minimizing this with respect to ΔX, gives

$$\frac{d\langle H\rangle}{d(\Delta X)} = \frac{-2\hbar^2}{8m(\Delta X)^3} + m\omega^2(\Delta X) = 0 \quad \Rightarrow \quad (\Delta X) = \left(\frac{\hbar}{2m\omega}\right)^{1/2}.$$

The resultant energy is

$$\langle H\rangle \ge \frac{\hbar^2}{8m(\hbar/2m\omega)} + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right) = \frac{1}{2}\hbar\omega\ ,$$

which is (miraculously) the correct energy.

2. a) For three particles with 2 possible single-particle states, there are $2^3 = 8$ possible three-particle states:

$$|aaa\rangle, |aab\rangle, |aba\rangle, |baa\rangle, |abb\rangle, |bab\rangle, |bba\rangle, |bbb\rangle$$

Symmetrizing these gives ($S = \frac{1}{6}(P_{123} + P_{132} + P_{213} + P_{231} + P_{312} + P_{321})$)

$$S|aaa\rangle = \tfrac{1}{6}(|aaa\rangle + |aaa\rangle + |aaa\rangle + |aaa\rangle + |aaa\rangle + |aaa\rangle) = |aaa\rangle$$
$$S|aab\rangle = \tfrac{1}{3}(|aab\rangle + |aba\rangle + |baa\rangle)$$
$$S|aba\rangle = \tfrac{1}{3}(|aab\rangle + |aba\rangle + |baa\rangle)$$
$$S|baa\rangle = \tfrac{1}{3}(|aab\rangle + |aba\rangle + |baa\rangle)$$
$$S|abb\rangle = \tfrac{1}{3}(|abb\rangle + |bab\rangle + |bba\rangle)$$
$$S|bab\rangle = \tfrac{1}{3}(|abb\rangle + |bab\rangle + |bba\rangle)$$
$$S|bba\rangle = \tfrac{1}{3}(|abb\rangle + |bab\rangle + |bba\rangle)$$
$$S|bbb\rangle = |bbb\rangle$$

There are 4 physically different states. Normalizing gives the states

$$|aaa,S\rangle = |aaa\rangle$$
$$|aab,S\rangle = \tfrac{1}{\sqrt{3}}(|aab\rangle + |aba\rangle + |baa\rangle)$$
$$|abb,S\rangle = \tfrac{1}{\sqrt{3}}(|abb\rangle + |bab\rangle + |bba\rangle)$$
$$|bbb,S\rangle = |bbb\rangle$$

For the antisymmetric states, we get ($A = \frac{1}{6}\left(P_{123} - P_{132} + P_{231} - P_{213} + P_{312} - P_{321}\right)$)

$$A|aaa\rangle = \tfrac{1}{6}\left(|aaa\rangle - |aaa\rangle + |aaa\rangle - |aaa\rangle + |aaa\rangle - |aaa\rangle\right) = 0$$
$$A|aab\rangle = \tfrac{1}{6}\left(|aab\rangle - |aba\rangle + |aba\rangle - |aab\rangle + |baa\rangle - |baa\rangle\right) = 0$$
$$A|aba\rangle = \tfrac{1}{6}\left(|aba\rangle - |aab\rangle + |baa\rangle - |baa\rangle + |aab\rangle - |aba\rangle\right) = 0$$
$$A|baa\rangle = \tfrac{1}{6}\left(|baa\rangle - |baa\rangle + |aab\rangle - |aba\rangle + |aba\rangle - |aab\rangle\right) = 0$$
$$A|abb\rangle = \tfrac{1}{6}\left(|abb\rangle - |abb\rangle + |bba\rangle - |bab\rangle + |bab\rangle - |bba\rangle\right) = 0$$
$$A|bab\rangle = \tfrac{1}{6}\left(|bab\rangle - |bba\rangle + |abb\rangle - |abb\rangle + |bba\rangle - |bab\rangle\right) = 0$$
$$A|bba\rangle = \tfrac{1}{6}\left(|bba\rangle - |bab\rangle + |bab\rangle - |bba\rangle + |abb\rangle - |abb\rangle\right) = 0$$
$$A|bbb\rangle = \tfrac{1}{6}\left(|bbb\rangle - |bbb\rangle + |bbb\rangle - |bbb\rangle + |bbb\rangle - |bbb\rangle\right) = 0$$

All of the states are null vectors, so there are 0 antisymmetric states.

b) There are 8 states in the direct product Hilbert space. The symmetric space has 4 states and the antisymmetric space has 0 states, so the number of states in the symmetric space is greater than the number of states in the antisymmetric space. Collectively, the symmetric space and the antisymmetric space (total 4 states) do not cover the direct product Hilbert space (total 8 states).

3. (a) Find $|\psi_\varepsilon\rangle = T(\varepsilon)|\psi\rangle$ in the position representation:

$$\begin{aligned}
\psi_\varepsilon(x) &= \langle x|\psi_\varepsilon\rangle = \langle x|T(\varepsilon)|\psi\rangle \\
&= \langle x|T(\varepsilon)\left\{\int |x'\rangle\langle x'|dx'\right\}|\psi\rangle \\
&= \int \langle x|T(\varepsilon)|x'\rangle\langle x'|\psi\rangle dx' \\
&= \int \langle x|x'+\varepsilon\rangle\langle x'|\psi\rangle dx' \\
&= \int \delta\left(x-(x'+\varepsilon)\right)\psi(x')dx' \\
&= \int \delta\left(x'-(x-\varepsilon)\right)\psi(x')dx' \\
&= \psi(x-\varepsilon)
\end{aligned}$$

Now use Taylor series expansion

$$\begin{aligned}
\langle x|T(\varepsilon)|\psi\rangle &= \psi(x-\varepsilon) \\
&= \psi(x) - \varepsilon \left.\frac{d\psi}{dx}\right|_x + \mathcal{O}\left(\varepsilon^2\right)
\end{aligned}$$

and put in the generator form;

$$\langle x|I - \frac{i}{\hbar}\varepsilon G|\psi\rangle = \psi(x) - \varepsilon\frac{d\psi}{dx}$$
$$\psi(x) - \frac{i}{\hbar}\varepsilon\langle x|G|\psi\rangle = \psi(x) - \varepsilon\frac{d\psi}{dx}$$
$$\Rightarrow \langle x|G|\psi\rangle = -i\hbar\frac{d\psi}{dx}$$

We know that the momentum operator in the position representation is

$$P \doteq -i\hbar \frac{d}{dx}$$

Hence we can conclude that

$$G = P$$

b) A system is translationally invariant if

$$\langle \psi | H | \psi \rangle = \langle \psi_\varepsilon | H | \psi_\varepsilon \rangle$$

Using the infinitesimal transformation

$$\begin{aligned}
\langle \psi | H | \psi \rangle &= \langle \psi_\varepsilon | H | \psi_\varepsilon \rangle \\
&= \langle T(\varepsilon)\psi_\varepsilon | H | T(\varepsilon)\psi_\varepsilon \rangle \\
&= \langle \psi_\varepsilon | T^\dagger(\varepsilon)\; H \; T(\varepsilon) | \psi \rangle
\end{aligned}$$

Now put in the generator form

$$\begin{aligned}
\langle \psi | H | \psi \rangle &= \langle \psi | T^\dagger(\varepsilon)\; H \; T(\varepsilon) | \psi \rangle \\
&= \langle \psi | \left(I - \frac{i\varepsilon}{\hbar} P \right)^\dagger H \left(I - \frac{i\varepsilon}{\hbar} P \right) | \psi \rangle \\
&= \langle \psi | \left(I + \frac{i\varepsilon}{\hbar} P \right) H \left(I - \frac{i\varepsilon}{\hbar} P \right) | \psi \rangle \\
&= \langle \psi | H - \frac{i\varepsilon}{\hbar} HP + \frac{i\varepsilon}{\hbar} PH + \frac{\varepsilon^2}{\hbar^2} PHP | \psi \rangle
\end{aligned}$$

Neglect the term of second order in the small quantity ε to get

$$\begin{aligned}
&\langle \psi | H | \psi \rangle = \langle \psi | H | \psi \rangle + \frac{i\varepsilon}{\hbar} \langle \psi | [P, H] | \psi \rangle \\
&\Rightarrow \langle \psi | [P, H] | \psi \rangle = \langle [P, H] \rangle = 0
\end{aligned}$$

Ehernfest's theorem applied here is

$$\frac{d}{dt} \langle P \rangle = \frac{1}{i\hbar} \langle [P, H] \rangle + \left\langle \frac{\partial P}{\partial t} \right\rangle$$

We assume that the operator P has no explicit time dependence, which eliminates the last term. Because translational invariance implies $\langle [P, H] \rangle = 0$, then the expectation value does not change with time (it is conserved):

$$\frac{d}{dt} \langle P \rangle = 0$$