13.1.5 Use Ehrenfest's theorem (Eq. 6.2)

$$\frac{d}{dt}\langle\Omega\rangle = \frac{-i}{\hbar}\langle[\Omega, H]\rangle$$

Assume a stationary state $|n\ell m\rangle$, so that $\langle\dot{\Omega}\rangle = 0$. Thus

$$\langle[\Omega, H]\rangle = 0$$

Use $\Omega = \mathbf{R}\bullet\mathbf{P}$ and note that the Hamiltonian is

$$H = \frac{P^2}{2m} - \frac{e^2}{R}$$

Find the commutator

$$\begin{aligned}
[\Omega, H] &= \left[\mathbf{R}\bullet\mathbf{P}, \frac{P^2}{2m} - \frac{e^2}{R}\right] = \left[\mathbf{R}\bullet\mathbf{P}, \frac{P^2}{2m}\right] - \left[\mathbf{R}\bullet\mathbf{P}, \frac{e^2}{R}\right] \\
&= \frac{1}{2m}\left[\mathbf{R}\bullet\mathbf{P}, P^2\right] - e^2\left[\mathbf{R}\bullet\mathbf{P}, \frac{1}{R}\right] = \frac{1}{2m}\left[\mathbf{R}, P^2\right]\bullet\mathbf{P} - e^2\mathbf{R}\bullet\left[\mathbf{P}, \frac{1}{R}\right] \\
&= \sum_{i=1}^{3}\frac{1}{2m}\left[R_i, P^2\right]P_i - e^2 R_i\left[P_i, \frac{1}{R}\right]
\end{aligned}$$

Now do each commutator:

$$\left[R_i, P^2\right] = \sum_{j=1}^{3}\left[R_i, P_j P_j\right] = \sum_{j=1}^{3}\left\{P_j\left[R_i, P_j\right] + \left[R_i, P_j\right]P_j\right\} = \sum_{j=1}^{3}\left\{P_j i\hbar\delta_{ij} + i\hbar\delta_{ij}P_j\right\} = 2i\hbar P_i$$

For the second commutator, work in the position representation

$$\begin{aligned}
\left[P_i, \frac{1}{R}\right]|\psi\rangle &\doteq \left[-i\hbar\frac{\partial}{\partial x_i}, \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right]\psi(r) \\
&= -i\hbar\left\{\frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\frac{\partial}{\partial x_i}\psi(r) + \psi(r)\frac{\partial}{\partial x_i}\frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} - \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\frac{\partial}{\partial x_i}\psi(r)\right\} \\
&\doteq -i\hbar\left\{\psi(r)\frac{-x_i}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}}\right\} = \left\{i\hbar\frac{x_i}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}}\right\}\psi(r) \\
&\Rightarrow \left[P_i, \frac{1}{R}\right] = i\hbar\frac{R_i}{R^3}
\end{aligned}$$

Put this together to get

$$[\Omega, H] = \sum_{i=1}^{3}\frac{1}{2m}2i\hbar P_i P_i - e^2 R_i i\hbar\frac{R_i}{R^3} = \frac{i\hbar}{m}P^2 - e^2 i\hbar\frac{1}{R} = i\hbar\left(\frac{P^2}{m} - \frac{e^2}{R}\right)$$

The expectation value must be zero, so

$$0=\left\langle \frac{P^2}{m}-\frac{e^2}{R}\right\rangle=\left\langle \frac{P^2}{m}\right\rangle-\left\langle \frac{e^2}{R}\right\rangle=2\left\langle \frac{P^2}{2m}\right\rangle+\left\langle \frac{-e^2}{R}\right\rangle=2\langle T\rangle+\langle V\rangle$$

Thus we get

$$\langle T\rangle=-\frac{1}{2}\langle V\rangle$$

14.3.6 The $\hat{\mathbf{n}}$ unit vector points in the direction defined by the polar angle θ from the *z*-axis and the azimuthal angle ϕ from the *x*-axis. We can imagine generating the $\hat{\mathbf{n}}$ unit vector by starting with the $\hat{\mathbf{z}}$ unit vector and rotating it away from the *z*-axis by the angle θ and then rotating around the the the *z*-axis by the angle ϕ. The first rotation should be around the *y*-axis in order to keep the vector in the *x*-*z* plane that defines $\phi=0$. If we apply this analysis to the spin up vector along the *z*-axis $|+\rangle=|s_z=+\hbar/2\rangle$, then this rotation will generate the spin up vector along the $\hat{\mathbf{n}}$ unit vector direction $|+\rangle_n=|s_n=+\hbar/2\rangle$. This description is represented by the operator equation

$$|+\rangle_n=U\left[R(\phi\hat{\mathbf{z}})\right]U\left[R(\theta\hat{\mathbf{y}})\right]|+\rangle$$

Let's now do the operation. The rotation matrices are

$$U\left[R(\vec{\theta})\right]=\cos\left(\frac{\theta}{2}\right)I-i\sin\left(\frac{\theta}{2}\right)\hat{\theta}\bullet\vec{\sigma}$$

$$U\left[R(\theta\hat{\mathbf{y}})\right]=\cos\left(\frac{\theta}{2}\right)I-i\sin\left(\frac{\theta}{2}\right)\sigma_y=\begin{pmatrix}\cos\frac{\theta}{2} & 0\\ 0 & \cos\frac{\theta}{2}\end{pmatrix}-i\begin{pmatrix}0 & -i\sin\frac{\theta}{2}\\ i\sin\frac{\theta}{2} & 0\end{pmatrix}=\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$$

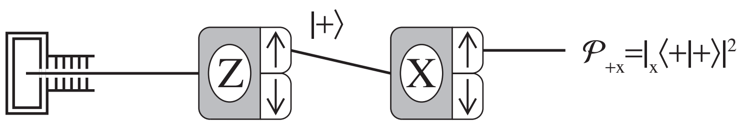
$$U\left[R(\phi\hat{\mathbf{z}})\right]=\cos\left(\frac{\phi}{2}\right)I-i\sin\left(\frac{\phi}{2}\right)\sigma_z=\begin{pmatrix}\cos\frac{\phi}{2} & 0\\ 0 & \cos\frac{\phi}{2}\end{pmatrix}-i\begin{pmatrix}\sin\frac{\phi}{2} & 0\\ 0 & -\sin\frac{\phi}{2}\end{pmatrix}=\begin{pmatrix}e^{-i\phi/2} & 0\\ 0 & e^{i\phi/2}\end{pmatrix}$$

Operate:

$$\begin{aligned}|+\rangle_n &\doteq \begin{pmatrix}e^{-i\phi/2} & 0\\ 0 & e^{i\phi/2}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &\doteq \begin{pmatrix}e^{-i\phi/2} & 0\\ 0 & e^{i\phi/2}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}\\ &\doteq \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\end{aligned}$$

As given in Eq. 14.3.28a.

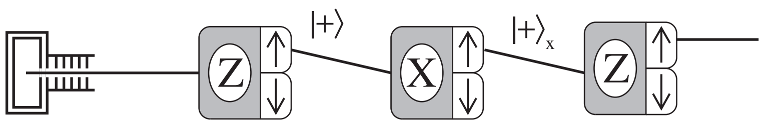
14.5.3 Expt 1:



Probability

$$\mathcal{P}_{out} = \left|\langle\psi_{out}|\psi_{in}\rangle\right|^2 = \left|{}_x\langle+|+\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

Expt 2:

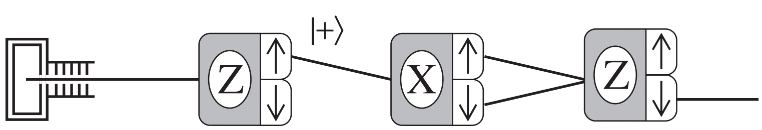


Probability

$$\mathcal{P}_{1\to3} = \mathcal{P}_{1\to2}\mathcal{P}_{2\to3} = \left|\langle\psi_{out2}|\psi_{out1\to in2}\rangle\right|^2\left|\langle\psi_{out3}|\psi_{out2=in3}\rangle\right|^2$$

$$= \left|{}_x\langle+|+\rangle\right|^2\left|\langle+|+\rangle_x\right|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right|^2\left|\begin{pmatrix}1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right|^2 = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$$

Expt 3:



Now we have a new situation at the 2nd analyzer. Both output ports are connected to the 3rd analyzer, which means that the probability of an atom from the 1st analyzer being input to the 3rd analyzer is 100%. So we need only calculate the probability of passage through the 3rd analyzer. The crucial step is determining the input state, for which we use the projection idea (p. 124). Because both states are used, the relevant projection operator is the sum of the two projection operators for each port, $P_{+x} + P_{-x}$, where $P_{+x} = |+\rangle_{x\,x}\langle+|$ and $P_{-x} = |-\rangle_{x\,x}\langle-|$. Thus the state after the 2nd analyzer is

$$\begin{aligned}\left|\psi_2\right\rangle &= \frac{\left(P_{+x}+P_{-x}\right)\left|\psi_1\right\rangle}{\sqrt{\left\langle\psi_1\right|\left(P_{+x}+P_{-x}\right)\left|\psi_1\right\rangle}} \\ &= \frac{\left(P_{+x}+P_{-x}\right)\left|+\right\rangle}{\sqrt{\left\langle+\right|\left(P_{+x}+P_{-x}\right)\left|+\right\rangle}}\end{aligned}$$

In this simple example, the projector $P_{+x}+P_{-x}$ is equal to the identity operator because the two states form a complete basis. This clearly simplifies the calculation, giving $\left|\psi_2\right\rangle=\left|+\right\rangle$, which means that the probability of exiting the 3rd analyzer is clearly 0. But to make sure our technique works, let's keep going. The denominator above equals one, giving

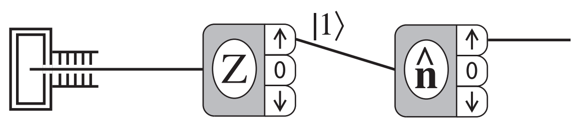
$$\begin{aligned}\left|\psi_2\right\rangle &= \left(\left|+\right\rangle_{x\,x}\left\langle+\right| + \left|-\right\rangle_{x\,x}\left\langle-\right|\right)\left|+\right\rangle \\ &= \left|+\right\rangle_{x\,x}\left\langle+|+\right\rangle + \left|-\right\rangle_{x\,x}\left\langle-|+\right\rangle\end{aligned}$$

Thus the beam entering the 3rd analyzer can be viewed as a **coherent** superposition of the eigenstates of the 2nd analyzer. Now calculate the probability of measuring spin down at the 3rd analyzer

$$\begin{aligned}\mathcal{P}_{1\to3} &= \mathcal{P}_{1\to2}\mathcal{P}_{2\to3} = (100\%)\left|\left\langle\psi_{out\,3}|\psi_{out\,2=in\,3}\right\rangle\right|^2 = \left|\left\langle-|\psi_{out\,2=in\,3}\right\rangle\right|^2 \\ &= \left|\left\langle-\right|\left(\left|+\right\rangle_{x\,x}\left\langle+|+\right\rangle + \left|-\right\rangle_{x\,x}\left\langle-|+\right\rangle\right)\right|^2 = \left|\left\langle-|+\right\rangle_{x\,x}\left\langle+|+\right\rangle + \left\langle-|-\right\rangle_{x\,x}\left\langle-|+\right\rangle\right|^2 \\ &= \left|\left\{\begin{pmatrix}0 & 1\end{pmatrix}\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\}\left\{\tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\} + \left\{\begin{pmatrix}0 & 1\end{pmatrix}\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\}\left\{\tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\}\right|^2 \\ &= \left|\left\{\frac{1}{\sqrt{2}}\right\}\left\{\frac{1}{\sqrt{2}}\right\} + \left\{\frac{-1}{\sqrt{2}}\right\}\left\{\frac{1}{\sqrt{2}}\right\}\right|^2 = \left|\frac{1}{2}-\frac{1}{2}\right|^2 \\ &= 0\end{aligned}$$

As expected.

14.5.4. Experiment:



For this problem we need to know the spin 1 operators (see McIntyre flyleaf).

$$S_z \doteq \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad S_x \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

The eigenstates of S_z are

$$|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |0\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |-1\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We also need to know the spin projection operator along the arbitrary direction $\hat{\mathbf{n}}$ defined by the polar angle θ from the z-axis and the azimuthal angle ϕ from the x-axis (see 14.3.27 for spin ½ case). This procedure applied to spin 1 gives

$$S_n \doteq \hbar\begin{pmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{2}}e^{-i\phi} & 0 \\ \frac{\sin\theta}{\sqrt{2}}e^{i\phi} & 0 & \frac{\sin\theta}{\sqrt{2}}e^{-i\phi} \\ 0 & \frac{\sin\theta}{\sqrt{2}}e^{i\phi} & -\cos\theta \end{pmatrix}$$

Diagonalize this to find the eigenstates:

$$|1\rangle_n = \frac{1+\cos\theta}{2}e^{-i\phi}|1\rangle + \frac{\sin\theta}{\sqrt{2}}|0\rangle + \frac{1-\cos\theta}{2}e^{i\phi}|-1\rangle$$
$$|0\rangle_n = -\frac{\sin\theta}{\sqrt{2}}e^{-i\phi}|1\rangle + \cos\theta|0\rangle + \frac{\sin\theta}{\sqrt{2}}e^{i\phi}|-1\rangle$$
$$|-1\rangle_n = \frac{1-\cos\theta}{2}e^{-i\phi}|1\rangle - \frac{\sin\theta}{\sqrt{2}}|0\rangle + \frac{1+\cos\theta}{2}e^{i\phi}|-1\rangle$$

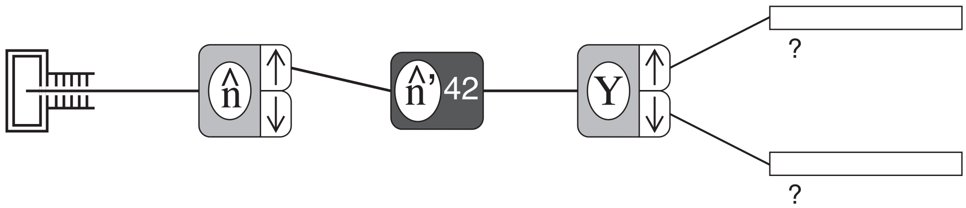
We need just the $|1\rangle_n$ state for the case $\phi = 0$:

$$|1\rangle_n = \frac{1+\cos\theta}{2}|1\rangle + \frac{\sin\theta}{\sqrt{2}}|0\rangle + \frac{1-\cos\theta}{2}|-1\rangle$$

Thus the probability is

$$\mathcal{P} = \left|\langle\psi_{out2}|\psi_{out1\to in2}\rangle\right|^2 = \left|{}_n\langle 1|1\rangle\right|^2$$
$$= \left|\begin{pmatrix} \frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right|^2 = \left(\frac{1+\cos\theta}{2}\right)^2 = \cos^4\frac{\theta}{2}$$

5. Experiment schematic:



The measurement at the first analyzer collapses the state to

$$|\psi(0)\rangle = |+\rangle_n = \cos\tfrac{\theta}{2}|+\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|-\rangle$$

The angles for the first analyzer are $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{4}$, so the initial state is

$$|\psi(0)\rangle = |+\rangle_n = \tfrac{1}{\sqrt{2}}|+\rangle + e^{i\pi/4}\tfrac{1}{\sqrt{2}}|-\rangle$$

In the field aligned at 45° in the xz-plane, the energy eigenstates are $|+\rangle_{n'} = \cos\frac{\theta'}{2}|+\rangle + \sin\frac{\theta'}{2}|-\rangle$ and $|-\rangle_{n'} = \sin\frac{\theta'}{2}|+\rangle - \cos\frac{\theta'}{2}|-\rangle$ with $\theta' = \pi/4$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. The initial state vector written in the energy basis is

$$\begin{aligned}|\psi(0)\rangle = |+\rangle_n &= \left(|+\rangle_{n'}\ {}_{n'}\langle+| + |-\rangle_{n'}\ {}_{n'}\langle-|\right)|+\rangle_n = \left({}_{n'}\langle+|+\rangle_n\right)|+\rangle_{n'} + \left({}_{n'}\langle-|+\rangle_n\right)|-\rangle_{n'}\\
&= \left(\tfrac{1}{\sqrt{2}}\cos\tfrac{\theta'}{2} + \tfrac{1}{\sqrt{2}}e^{i\pi/4}\sin\tfrac{\theta'}{2}\right)|+\rangle_{n'} + \left(\tfrac{1}{\sqrt{2}}\sin\tfrac{\theta'}{2} - \tfrac{1}{\sqrt{2}}e^{i\pi/4}\cos\tfrac{\theta'}{2}\right)|-\rangle_{n'}\end{aligned}$$

The time evolved state is

$$\begin{aligned}|\psi(t)\rangle &= \left(\tfrac{1}{\sqrt{2}}\cos\tfrac{\theta'}{2} + \tfrac{1}{\sqrt{2}}e^{i\pi/4}\sin\tfrac{\theta'}{2}\right)e^{-iE_+t/\hbar}|+\rangle_{n'} + \left(\tfrac{1}{\sqrt{2}}\sin\tfrac{\theta'}{2} - \tfrac{1}{\sqrt{2}}e^{i\pi/4}\cos\tfrac{\theta'}{2}\right)e^{-iE_-t/\hbar}|-\rangle_{n'}\\
&= \left(\tfrac{1}{\sqrt{2}}\cos\tfrac{\theta'}{2} + \tfrac{1}{\sqrt{2}}e^{i\pi/4}\sin\tfrac{\theta'}{2}\right)e^{-i\omega_0 t/2}|+\rangle_{n'} + \left(\tfrac{1}{\sqrt{2}}\sin\tfrac{\theta'}{2} - \tfrac{1}{\sqrt{2}}e^{i\pi/4}\cos\tfrac{\theta'}{2}\right)e^{+i\omega_0 t/2}|-\rangle_{n'}\end{aligned}$$

The probability of measuring S_y to be $+\hbar/2$ is

$$\mathcal{P}_{+y} = \left|\langle\psi_{out}|\psi_{in}\rangle\right|^2 = \left|{}_y\langle+|\psi(t)\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\left\{\begin{array}{l}\left(\tfrac{1}{\sqrt{2}}\cos\tfrac{\theta'}{2}+\tfrac{1}{\sqrt{2}}e^{i\pi/4}\sin\tfrac{\theta'}{2}\right)e^{-i\omega_0 t/2}\begin{pmatrix}\cos\tfrac{\theta'}{2}\\ \sin\tfrac{\theta'}{2}\end{pmatrix}\\ +\left(\tfrac{1}{\sqrt{2}}\sin\tfrac{\theta'}{2}-\tfrac{1}{\sqrt{2}}e^{i\pi/4}\cos\tfrac{\theta'}{2}\right)e^{+i\omega_0 t/2}\begin{pmatrix}\sin\tfrac{\theta'}{2}\\ -\cos\tfrac{\theta'}{2}\end{pmatrix}\end{array}\right\}\right|^2$$

$$= \tfrac{1}{4}\left|\begin{array}{l}\left(\cos\tfrac{\theta'}{2}+e^{i\pi/4}\sin\tfrac{\theta'}{2}\right)\left(\cos\tfrac{\theta'}{2}-i\sin\tfrac{\theta'}{2}\right)e^{-i\omega_0 t/2}+\\ +\left(\sin\tfrac{\theta'}{2}-e^{i\pi/4}\cos\tfrac{\theta'}{2}\right)\left(\sin\tfrac{\theta'}{2}+i\cos\tfrac{\theta'}{2}\right)e^{+i\omega_0 t/2}\end{array}\right|^2$$

$$= \tfrac{1}{4}\left|\begin{array}{l}\cos^2\tfrac{\theta'}{2}e^{-i\omega_0 t/2}+\sin^2\tfrac{\theta'}{2}e^{+i\omega_0 t/2}-2\sin\tfrac{\theta'}{2}\cos\tfrac{\theta'}{2}\sin\tfrac{\omega_0 t}{2}+\\ -2ie^{i\pi/4}\sin\tfrac{\theta'}{2}\cos\tfrac{\theta'}{2}\sin\tfrac{\omega_0 t}{2}-ie^{i\pi/4}\sin^2\tfrac{\theta'}{2}e^{-i\omega_0 t/2}-ie^{i\pi/4}\cos^2\tfrac{\theta'}{2}e^{+i\omega_0 t/2}\end{array}\right|^2$$

$$= \tfrac{1}{4}\left|\begin{array}{l}\left(1+ie^{i\pi/4}\right)e^{+i\omega_0 t/2}-2i\left(1+ie^{i\pi/4}\right)\cos^2\tfrac{\theta'}{2}\sin\tfrac{\omega_0 t}{2}+\\ -2ie^{i\pi/4}\cos\tfrac{\omega_0 t}{2}-\left(1+ie^{i\pi/4}\right)\sin\theta'\sin\tfrac{\omega_0 t}{2}\end{array}\right|^2$$

$$= \tfrac{1}{4}\left|\begin{array}{l}\left(\tfrac{\sqrt{2}-1}{\sqrt{2}}+i\tfrac{1}{\sqrt{2}}\right)\left(\cos\tfrac{\omega_0 t}{2}+i\sin\tfrac{\omega_0 t}{2}\right)+\left(\tfrac{1}{\sqrt{2}}-i\tfrac{\sqrt{2}-1}{\sqrt{2}}\right)\left(1+\cos\theta'\right)\sin\tfrac{\omega_0 t}{2}+\\ +\left(\tfrac{1-i}{\sqrt{2}}\right)2\cos\tfrac{\omega_0 t}{2}-\left(\tfrac{\sqrt{2}-1}{\sqrt{2}}+i\tfrac{1}{\sqrt{2}}\right)\sin\theta'\sin\tfrac{\omega_0 t}{2}\end{array}\right|^2$$

$$= \tfrac{1}{8}\left|\begin{array}{l}\left\{\left(1+\sqrt{2}\right)\cos\tfrac{\omega_0 t}{2}+\left(\cos\theta'-\left(\sqrt{2}-1\right)\sin\theta'\right)\sin\tfrac{\omega_0 t}{2}\right\}+\\ +i\left\{-\cos\tfrac{\omega_0 t}{2}-\left(\left(\sqrt{2}+1\right)\cos\theta'+\sin\theta'\right)\sin\tfrac{\omega_0 t}{2}\right\}\end{array}\right|^2$$

Squaring and simplifying gives

$$\begin{aligned}\mathcal{P}_{+y} &= \tfrac{1}{8}\left\{\left(4+2\sqrt{2}\right)\cos^2\tfrac{\omega_0 t}{2}+\left(4-2\sqrt{2}\right)\sin^2\tfrac{\omega_0 t}{2}+4\sqrt{2}\cos\theta'\cos\tfrac{\omega_0 t}{2}\sin\tfrac{\omega_0 t}{2}\right\}\\ &= \tfrac{1}{4}\left\{2+\sqrt{2}\cos\omega_0 t+\sqrt{2}\cos\theta'\sin\omega_0 t\right\}\\ &= \tfrac{1}{4}\left\{2+\sqrt{2}\cos\omega_0 t+\sin\omega_0 t\right\}\\ &= \tfrac{1}{2}+\tfrac{1}{2\sqrt{2}}\cos\omega_0 t+\tfrac{1}{4}\sin\omega_0 t = \tfrac{1}{2}+\tfrac{\sqrt{3}}{4}\cos\left(\omega_0 t-\tan^{-1}\left[1/\sqrt{2}\right]\right)\end{aligned}$$