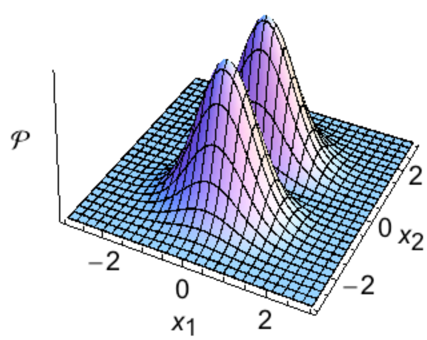
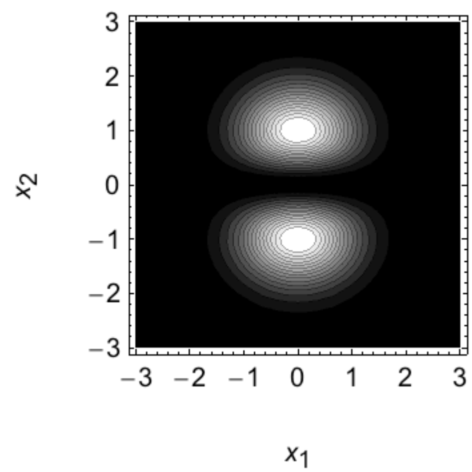
1. The 2-particle wave function is a product of two single-particle wave functions. For the single-particle wave functions we use the dimensionless harmonic oscillator wave functions:

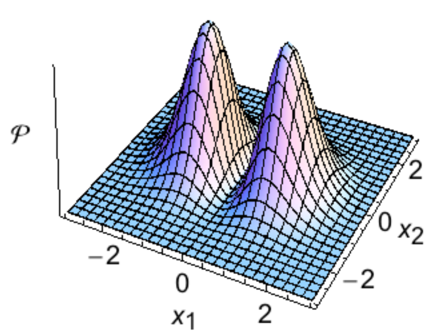
$$\begin{aligned} \varphi_0(x) &= \frac{1}{\pi^{1/4}} e^{-x^2/2} \\ \varphi_1(x) &= \frac{\sqrt{2}}{\pi^{1/4}} x e^{-x^2/2} \end{aligned}.$$

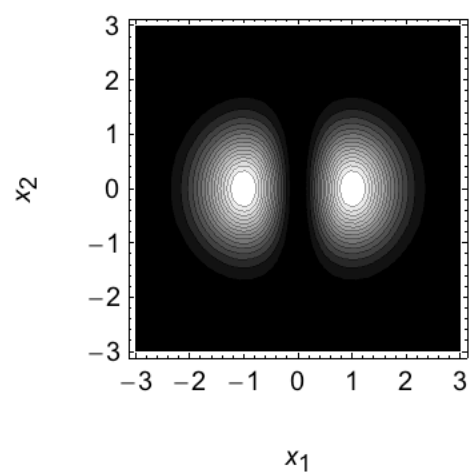
The following plots for the probability density result.

(a) Distinguishable particles: We could either have $n_a = 0$ and $n_b = 1$ or $n_a = 1$ and $n_b = 0$, which are different in this case:



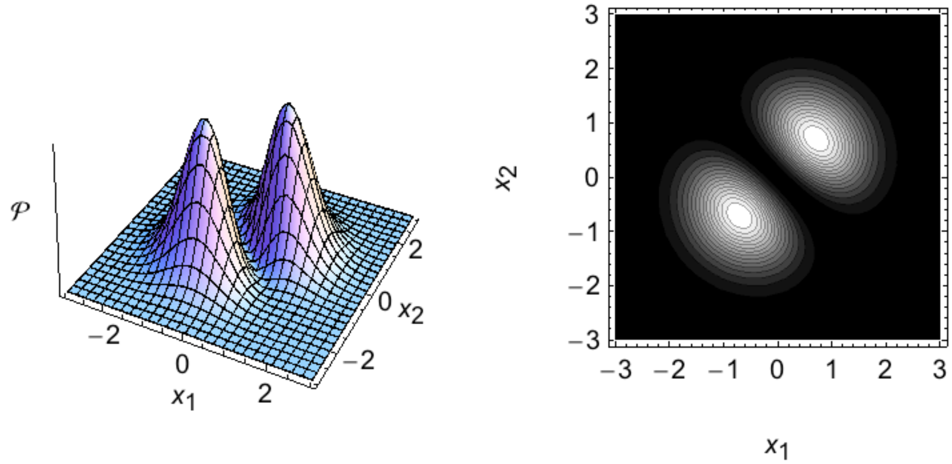






(b) For a symmetric spatial state, the wave function is

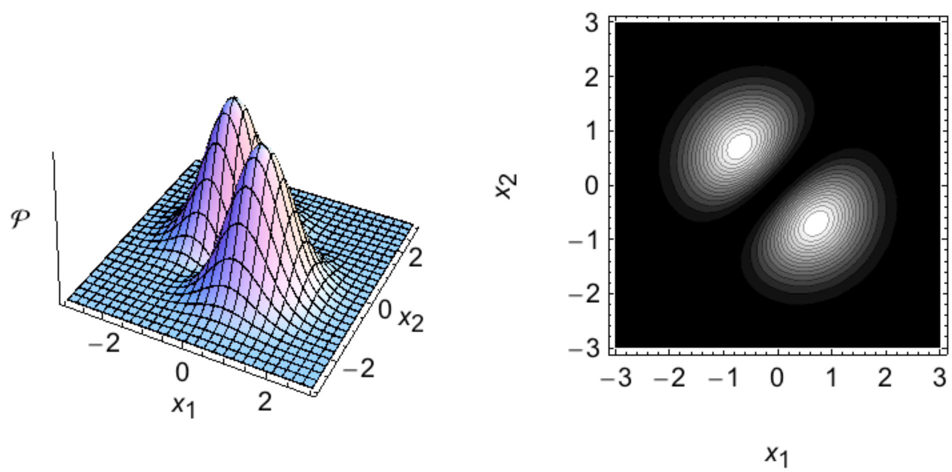
$$\psi^S_{n_a=0,n_b=1}(x_1,x_2) = \tfrac{1}{\sqrt{2}}\left[\varphi_0(x_1)\varphi_1(x_2) + \varphi_0(x_2)\varphi_1(x_1)\right]$$



Notice that the line $x_1 = x_2$ crosses regions of very high probability amplitude. In other words, if you asked the question, "What is the probability that particle #1 is in some interval around, say, $x_1 = 1/3$, subject to particle #2 *also* being in that interval?", the answer would be "quite good", *regardless* of what position in the box you asked the question about. (Obviously not about $x_1 = 0$, but this is a node in one of the single-particle wave functions). Particles in a symmetric spatial state tend to seek each other out *even though they are "non-interacting" particles*.

(c) For an antisymmetric spatial state, the wave function is:

$$\psi^A_{n_a=0,n_b=1}(x_1,x_2) = \tfrac{1}{\sqrt{2}}\left[\varphi_0(x_1)\varphi_1(x_2) - \varphi_0(x_2)\varphi_1(x_1)\right]$$



Notice that the line $x_1 = x_2$ falls in a region of very low probability amplitude. In other words, if you asked the question, "What is the probability that particle #1 is in some interval around, say, $x_1 = 1/3$, subject to particle #2 *also* being in that interval?", the answer would be "very small", *regardless* of what position in the box you asked the question about. Particles in an antisymmetric spatial state tend to avoid each other *even though they are "non-interacting" particles*.

2. For the first excited state of the harmonic oscillator system with one particle in the single-particle ground state and one particle in the first single-particle excited state, the probability densities for the distinguishable (D) particle case, the symmetric (S) identical particle case and the antisymmetric (A) case are

$$\mathcal{P}_D(x_1,x_2) = \left|\varphi_0(x_1)\varphi_1(x_2)\right|^2 = \left(\frac{1}{\pi^{1/4}} e^{-x_1^2/2}\right)^2 \left(\frac{\sqrt{2}}{\pi^{1/4}} x_2\, e^{-x_2^2/2}\right)^2$$

$$\begin{aligned}\mathcal{P}_S(x_1,x_2) &= \left|\frac{1}{\sqrt{2}}\left[\varphi_0(x_1)\varphi_1(x_2) + \varphi_0(x_2)\varphi_1(x_1)\right]\right|^2 \\ &= \frac{1}{2}\left[\left(\frac{1}{\pi^{1/4}} e^{-x_1^2/2}\right)\left(\frac{\sqrt{2}}{\pi^{1/4}} x_2\, e^{-x_2^2/2}\right) + \left(\frac{1}{\pi^{1/4}} e^{-x_2^2/2}\right)\left(\frac{\sqrt{2}}{\pi^{1/4}} x_1\, e^{-x_1^2/2}\right)\right]^2\end{aligned}$$

$$\begin{aligned}\mathcal{P}_A(x_1,x_2) &= \left|\frac{1}{\sqrt{2}}\left[\varphi_0(x_1)\varphi_1(x_2) - \varphi_0(x_2)\varphi_1(x_1)\right]\right|^2 \\ &= \frac{1}{2}\left[\left(\frac{1}{\pi^{1/4}} e^{-x_1^2/2}\right)\left(\frac{\sqrt{2}}{\pi^{1/4}} x_2\, e^{-x_2^2/2}\right) - \left(\frac{1}{\pi^{1/4}} e^{-x_2^2/2}\right)\left(\frac{\sqrt{2}}{\pi^{1/4}} x_1\, e^{-x_1^2/2}\right)\right]^2\end{aligned}$$

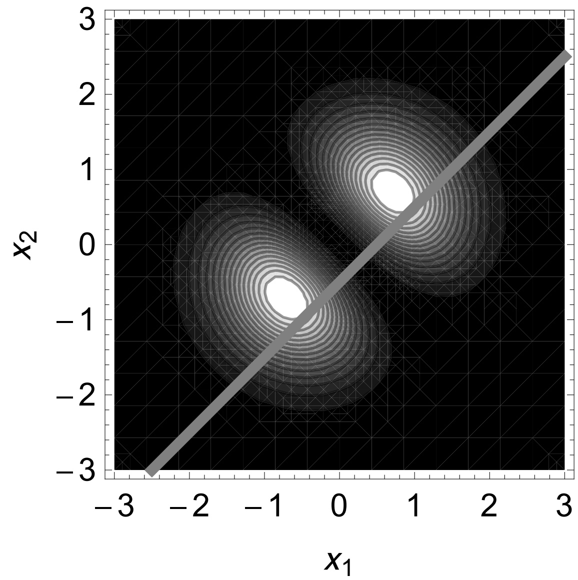
where we have used dimensionless variables. To find the particle separation probability density $\mathcal{P}(x_1 - x_2)$, make a change of variables from the individual coordinates x_1 and x_2 to the center-of-mass and relative coordinates.

$$\begin{aligned} X_{CM} &= \frac{x_1+x_2}{2} \equiv u \\ x_{rel} &= (x_1 - x_2) \equiv v \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= \frac{2u+v}{2} \\ x_2 &= \frac{2u-v}{2} \end{aligned}$$

and then integrate out the center-of-mass coordinate u. The transformed probability densities are

$$\begin{aligned}\mathcal{P}_D(u,v) &= \frac{1}{2\pi}\left[e^{-u^2} e^{-v^2/4}(2u-v)\right]^2 = \frac{1}{2\pi} e^{-2u^2} e^{-v^2/2}(2u-v)^2 \\ \mathcal{P}_S(u,v) &= \frac{1}{4\pi}\left[e^{-u^2} e^{-v^2/4}(2u-v) + e^{-u^2} e^{-v^2/4}(2u+v)\right]^2 = \frac{4}{\pi} u^2 e^{-2u^2} e^{-v^2/2} \\ \mathcal{P}_A(u,v) &= \frac{1}{4\pi}\left[e^{-u^2} e^{-v^2/4}(2u-v) - e^{-u^2} e^{-v^2/4}(2u+v)\right]^2 = \frac{1}{\pi} v^2 e^{-2u^2} e^{-v^2/2}\end{aligned}$$

The integration is along a line for which $v = x_1 - x_2 = \text{constant}$. For example, along a line going through the points (-2.5, -3) and (3, 2.5), as shown below.



In terms of the transformed coordinates, this integral is from $u=-\infty$ to $u=+\infty$. For the distinguishable case, the integration gives

$$\begin{aligned}
\mathcal{P}_D(v) &= \int_{-\infty}^{\infty} \mathcal{P}_D(u,v)\,du = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-2u^2} e^{-v^2/2} (2u-v)^2\, du \\
&= \frac{1}{2\pi} e^{-v^2/2}\left[4\int_{-\infty}^{\infty} u^2 e^{-2u^2}\, du - 4v\int_{-\infty}^{\infty} u e^{-2u^2}\, du + v^2 \int_{-\infty}^{\infty} e^{-2u^2}\, du\right] \\
&= \frac{1}{2\pi} e^{-v^2/2}\left[4\left(\frac{1}{4}\sqrt{\frac{\pi}{2}}\right) - 4v(0) + v^2\left(\sqrt{\frac{\pi}{2}}\right)\right] \\
&= \frac{1}{2\sqrt{2\pi}} e^{-v^2/2}\left(1+v^2\right)
\end{aligned}$$

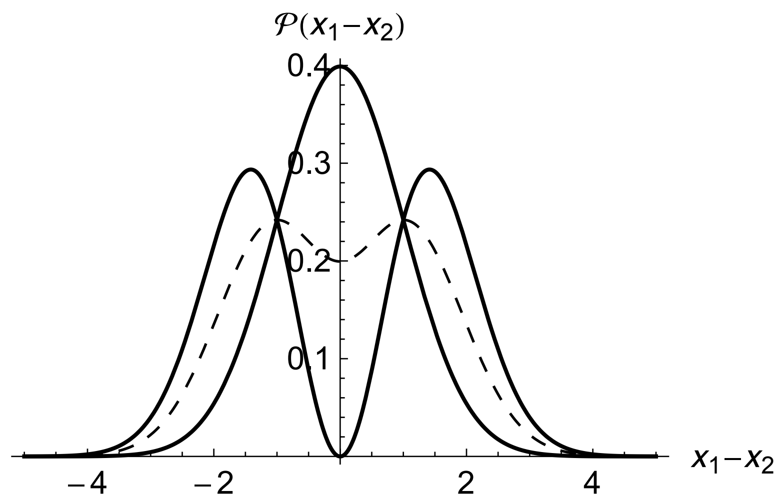
For the identical particle cases, we get

$$\begin{aligned}
\mathcal{P}_S(v) &= \int_{-\infty}^{\infty} \mathcal{P}_S(u,v)\,du = \int_{-\infty}^{\infty} \frac{4}{\pi} u^2 e^{-2u^2} e^{-v^2/2}\, du \\
&= \frac{4}{\pi} e^{-v^2/2} \int_{-\infty}^{\infty} u^2 e^{-2u^2}\, du \\
&= \frac{4}{\pi} e^{-v^2/2}\left(\frac{1}{4}\sqrt{\frac{\pi}{2}}\right) \\
&= \frac{1}{\sqrt{2\pi}} e^{-v^2/2}
\end{aligned}$$

and

$$\mathcal{P}_A(v) = \int_{-\infty}^{\infty} \mathcal{P}_S(u,v)\,du = \int_{-\infty}^{\infty} \frac{1}{\pi} v^2 e^{-2u^2} e^{-v^2/2}\,du$$
$$= \frac{1}{\pi} v^2 e^{-v^2/2} \int_{-\infty}^{\infty} e^{-2u^2}\,du$$
$$= \frac{1}{\pi} v^2 e^{-v^2/2} \left(\sqrt{\frac{\pi}{2}} \right)$$
$$= \frac{1}{\sqrt{2\pi}} v^2 e^{-v^2/2}$$

Plot: The distinguishable particle case is shown dashed. The symmetric case is peaked at the origin, indicating the tendency of particles in a symmetric spatial state to "attract" each other. The antisymmetric case is zero at the origin, indicating the tendency of particles in an antisymmetric spatial state to "repel" each other. Note that each of these integrates to unity as expected. One could argue, as in the text, that we should use only the positive $v = x_1 - x_2$ part (or absolute value) of these probability densities and multiply by 2.



3. a) Observable Ω commutes with the Hamiltonian ($[\Omega,H]=0$). Find the time derivative of the expectation value:

$$\frac{d}{dt}\langle\Omega\rangle=\frac{d}{dt}\langle\psi|\Omega|\psi\rangle=\left(\frac{d}{dt}\langle\psi|\right)\Omega|\psi\rangle+\langle\psi|\Omega\left(\frac{d}{dt}|\psi\rangle\right)+\langle\psi|\frac{\partial}{\partial t}\Omega|\psi\rangle$$

Use the Schrodinger equation and its adjoint to get

$$\begin{aligned}\frac{d}{dt}\langle\Omega\rangle&=\left(-\frac{1}{i\hbar}\langle\psi|H\right)\Omega|\psi\rangle+\langle\psi|\Omega\left(\frac{1}{i\hbar}H|\psi\rangle\right)+\langle\psi|\frac{\partial}{\partial t}\Omega|\psi\rangle\\&=\frac{1}{i\hbar}\langle\psi|(\Omega H-H\Omega)|\psi\rangle+\langle\psi|\frac{\partial}{\partial t}\Omega|\psi\rangle\\&=\frac{1}{i\hbar}\langle[\Omega,H]\rangle+\left\langle\frac{\partial\Omega}{\partial t}\right\rangle\end{aligned}$$

We assume that the operator Ω has no explicit time dependence, which eliminates the last term. Hence, if the observable Ω commutes with the Hamiltonian ($[\Omega,H]=0$), then the expectation value does not change with time (it is conserved).

b) If A is unchanged by a unitary transformation U, then

$$U^\dagger AU=A$$

Act on both sides with U and use the unitary condition $UU^\dagger=I$

$$\begin{aligned}UU^\dagger AU&=UA\\IAU&=UA\\AU-UA&=0\\[A,U]&=0\end{aligned}$$

So U and A commute.

c) Do the same, but now with the infinitesimal form, noting that G is Hermitian:

$$\begin{aligned}U^\dagger AU&=A\\\left(I-\frac{i\varepsilon}{\hbar}G\right)^\dagger A\left(I-\frac{i\varepsilon}{\hbar}G\right)&=A\\\left(I+\frac{i\varepsilon}{\hbar}G\right)A\left(I-\frac{i\varepsilon}{\hbar}G\right)&=A\\A-\frac{i\varepsilon}{\hbar}AG+\frac{i\varepsilon}{\hbar}GA+\frac{\varepsilon^2}{\hbar^2}GAG&=A\end{aligned}$$

Neglect the term of second order in the small quantity ε to get

$$\begin{aligned}\frac{i\varepsilon}{\hbar}(GA-AG)&=0\\[G,A]&=0\end{aligned}$$

So G and A commute.