10.1.2 Single-particle basis vectors are $|+\rangle$ and $|-\rangle$. Two-particle basis vectors are $|++\rangle$, $|+-\rangle$, $|-+\rangle$ and $|--\rangle$. Single-particle operators are

$$\sigma_1^{(1)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \sigma_2^{(2)} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

with row/column labeling of + and -. Now find two-particle operators, using labeling ++, +-, -+, --. First do $\sigma_1^{(1)\otimes(2)} = \sigma_1^{(1)} \otimes I^{(2)}$. For the first row, we get

$$\langle ++|\sigma_1^{(1)\otimes(2)}|++\rangle = \langle ++|\sigma_1^{(1)} \otimes I^{(2)}|++\rangle = \langle +|\sigma_1^{(1)}|+\rangle\langle +|I^{(2)}|+\rangle = a*1 = a$$
$$\langle ++|\sigma_1^{(1)\otimes(2)}|+-\rangle = \langle ++|\sigma_1^{(1)} \otimes I^{(2)}|+-\rangle = \langle +|\sigma_1^{(1)}|+\rangle\langle +|I^{(2)}|-\rangle = a*0 = 0$$
$$\langle ++|\sigma_1^{(1)\otimes(2)}|-+\rangle = \langle ++|\sigma_1^{(1)} \otimes I^{(2)}|-+\rangle = \langle +|\sigma_1^{(1)}|-\rangle\langle +|I^{(2)}|+\rangle = b*1 = b$$
$$\langle ++|\sigma_1^{(1)\otimes(2)}|--\rangle = \langle ++|\sigma_1^{(1)} \otimes I^{(2)}|--\rangle = \langle +|\sigma_1^{(1)}|-\rangle\langle +|I^{(2)}|-\rangle = b*0 = 0$$

For the second row, we get

$$\langle +-|\sigma_1^{(1)\otimes(2)}|++\rangle = \langle +-|\sigma_1^{(1)} \otimes I^{(2)}|++\rangle = \langle +|\sigma_1^{(1)}|+\rangle\langle -|I^{(2)}|+\rangle = a*0 = 0$$
$$\langle +-|\sigma_1^{(1)\otimes(2)}|+-\rangle = \langle +-|\sigma_1^{(1)} \otimes I^{(2)}|+-\rangle = \langle +|\sigma_1^{(1)}|+\rangle\langle -|I^{(2)}|-\rangle = a*1 = a$$
$$\langle +-|\sigma_1^{(1)\otimes(2)}|-+\rangle = \langle +-|\sigma_1^{(1)} \otimes I^{(2)}|-+\rangle = \langle +|\sigma_1^{(1)}|-\rangle\langle -|I^{(2)}|+\rangle = b*0 = 0$$
$$\langle +-|\sigma_1^{(1)\otimes(2)}|--\rangle = \langle +-|\sigma_1^{(1)} \otimes I^{(2)}|--\rangle = \langle +|\sigma_1^{(1)}|-\rangle\langle -|I^{(2)}|-\rangle = b*1 = b$$

For the third row, we get

$$\langle -+|\sigma_1^{(1)\otimes(2)}|++\rangle = \langle -+|\sigma_1^{(1)} \otimes I^{(2)}|++\rangle = \langle -|\sigma_1^{(1)}|+\rangle\langle +|I^{(2)}|+\rangle = c*1 = c$$
$$\langle -+|\sigma_1^{(1)\otimes(2)}|+-\rangle = \langle -+|\sigma_1^{(1)} \otimes I^{(2)}|+-\rangle = \langle -|\sigma_1^{(1)}|+\rangle\langle +|I^{(2)}|-\rangle = c*0 = 0$$
$$\langle -+|\sigma_1^{(1)\otimes(2)}|-+\rangle = \langle -+|\sigma_1^{(1)} \otimes I^{(2)}|-+\rangle = \langle -|\sigma_1^{(1)}|-\rangle\langle +|I^{(2)}|+\rangle = d*1 = d$$
$$\langle -+|\sigma_1^{(1)\otimes(2)}|--\rangle = \langle -+|\sigma_1^{(1)} \otimes I^{(2)}|--\rangle = \langle -|\sigma_1^{(1)}|-\rangle\langle +|I^{(2)}|-\rangle = d*0 = 0$$

For the fourth row, we get

$$\langle --|\sigma_1^{(1)\otimes(2)}|++\rangle = \langle --|\sigma_1^{(1)} \otimes I^{(2)}|++\rangle = \langle -|\sigma_1^{(1)}|+\rangle\langle -|I^{(2)}|+\rangle = c*0 = 0$$
$$\langle --|\sigma_1^{(1)\otimes(2)}|+-\rangle = \langle --|\sigma_1^{(1)} \otimes I^{(2)}|+-\rangle = \langle -|\sigma_1^{(1)}|+\rangle\langle -|I^{(2)}|-\rangle = c*1 = c$$
$$\langle --|\sigma_1^{(1)\otimes(2)}|-+\rangle = \langle --|\sigma_1^{(1)} \otimes I^{(2)}|-+\rangle = \langle -|\sigma_1^{(1)}|-\rangle\langle -|I^{(2)}|+\rangle = d*0 = 0$$
$$\langle --|\sigma_1^{(1)\otimes(2)}|--\rangle = \langle --|\sigma_1^{(1)} \otimes I^{(2)}|--\rangle = \langle -|\sigma_1^{(1)}|-\rangle\langle -|I^{(2)}|-\rangle = d*1 = d$$

The result is

$$\sigma_1^{(1)\otimes(2)} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

Now do $\sigma_2^{(1)\otimes(2)} = I^{(1)} \otimes \sigma_2^{(2)}$. For the first row, we get

$$\langle ++|\sigma_2^{(1)\otimes(2)}|++\rangle = \langle ++|I^{(1)} \otimes \sigma_2^{(2)}|++\rangle = \langle +|I^{(1)}|+\rangle\langle +|\sigma_2^{(2)}|+\rangle = 1*e = e$$
$$\langle ++|\sigma_2^{(1)\otimes(2)}|+-\rangle = \langle ++|I^{(1)} \otimes \sigma_2^{(2)}|+-\rangle = \langle +|I^{(1)}|+\rangle\langle +|\sigma_2^{(2)}|-\rangle = 1*f = f$$
$$\langle ++|\sigma_2^{(1)\otimes(2)}|-+\rangle = \langle ++|I^{(1)} \otimes \sigma_2^{(2)}|-+\rangle = \langle +|I^{(1)}|-\rangle\langle +|\sigma_2^{(2)}|+\rangle = 0*e = 0$$
$$\langle ++|\sigma_2^{(1)\otimes(2)}|--\rangle = \langle ++|I^{(1)} \otimes \sigma_2^{(2)}|--\rangle = \langle +|I^{(1)}|-\rangle\langle +|\sigma_2^{(2)}|-\rangle = 0*f = 0$$

For the second row, we get

$$\langle +-|\sigma_2^{(1)\otimes(2)}|++\rangle = \langle +-|I^{(1)}\otimes\sigma_2^{(2)}|++\rangle = \langle +|I^{(1)}|+\rangle\langle -|\sigma_2^{(2)}|+\rangle = 1*g = g$$
$$\langle +-|\sigma_2^{(1)\otimes(2)}|+-\rangle = \langle +-|I^{(1)}\otimes\sigma_2^{(2)}|+-\rangle = \langle +|I^{(1)}|+\rangle\langle -|\sigma_2^{(2)}|-\rangle = 1*h = h$$
$$\langle +-|\sigma_2^{(1)\otimes(2)}|-+\rangle = \langle +-|I^{(1)}\otimes\sigma_2^{(2)}|-+\rangle = \langle +|I^{(1)}|-\rangle\langle -|\sigma_2^{(2)}|+\rangle = 0*g = 0$$
$$\langle +-|\sigma_2^{(1)\otimes(2)}|--\rangle = \langle +-|I^{(1)}\otimes\sigma_2^{(2)}|--\rangle = \langle +|I^{(1)}|-\rangle\langle -|\sigma_2^{(2)}|-\rangle = 0*h = 0$$

For the third row, we get

$$\langle -+|\sigma_2^{(1)\otimes(2)}|++\rangle = \langle -+|I^{(1)}\otimes\sigma_2^{(2)}|++\rangle = \langle -|I^{(1)}|+\rangle\langle +|\sigma_2^{(2)}|+\rangle = 0*e = 0$$
$$\langle -+|\sigma_2^{(1)\otimes(2)}|+-\rangle = \langle -+|I^{(1)}\otimes\sigma_2^{(2)}|+-\rangle = \langle -|I^{(1)}|+\rangle\langle +|\sigma_2^{(2)}|-\rangle = 0*f = 0$$
$$\langle -+|\sigma_2^{(1)\otimes(2)}|-+\rangle = \langle -+|I^{(1)}\otimes\sigma_2^{(2)}|-+\rangle = \langle -|I^{(1)}|-\rangle\langle +|\sigma_2^{(2)}|+\rangle = 1*e = e$$
$$\langle -+|\sigma_2^{(1)\otimes(2)}|--\rangle = \langle -+|I^{(1)}\otimes\sigma_2^{(2)}|--\rangle = \langle -|I^{(1)}|-\rangle\langle +|\sigma_2^{(2)}|-\rangle = 1*f = f$$

For the fourth row, we get

$$\langle --|\sigma_2^{(1)\otimes(2)}|++\rangle = \langle --|I^{(1)}\otimes\sigma_2^{(2)}|++\rangle = \langle -|I^{(1)}|+\rangle\langle -|\sigma_2^{(2)}|+\rangle = 0*g = 0$$
$$\langle --|\sigma_2^{(1)\otimes(2)}|+-\rangle = \langle --|I^{(1)}\otimes\sigma_2^{(2)}|+-\rangle = \langle -|I^{(1)}|+\rangle\langle -|\sigma_2^{(2)}|-\rangle = 0*h = 0$$
$$\langle --|\sigma_2^{(1)\otimes(2)}|-+\rangle = \langle --|I^{(1)}\otimes\sigma_2^{(2)}|-+\rangle = \langle -|I^{(1)}|-\rangle\langle -|\sigma_2^{(2)}|+\rangle = 1*g = g$$
$$\langle --|\sigma_2^{(1)\otimes(2)}|--\rangle = \langle --|I^{(1)}\otimes\sigma_2^{(2)}|--\rangle = \langle -|I^{(1)}|-\rangle\langle -|\sigma_2^{(2)}|-\rangle = 1*h = h$$

The result is

$$\sigma_2^{(1)\otimes(2)} = \begin{pmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix}$$

Now find $(\sigma_1\sigma_2)^{(1)\otimes(2)} = \sigma_1^{(1)}\otimes\sigma_2^{(2)}$. Do first by simple product of $\sigma_1^{(1)\otimes(2)}$ and $\sigma_2^{(1)\otimes(2)}$:

$$(\sigma_1\sigma_2)^{(1)\otimes(2)} = \sigma_1^{(1)\otimes(2)}\sigma_2^{(1)\otimes(2)} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}\begin{pmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

Now find with matrix elements. First row:

$$\langle ++|(\sigma_1\sigma_2)^{(1)\otimes(2)}|++\rangle = \langle ++|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|++\rangle = \langle +|\sigma_1^{(1)}|+\rangle\langle +|\sigma_2^{(2)}|+\rangle = a*e = ae$$
$$\langle ++|(\sigma_1\sigma_2)^{(1)\otimes(2)}|+-\rangle = \langle ++|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|+-\rangle = \langle +|\sigma_1^{(1)}|+\rangle\langle +|\sigma_2^{(2)}|-\rangle = a*f = af$$
$$\langle ++|(\sigma_1\sigma_2)^{(1)\otimes(2)}|-+\rangle = \langle ++|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|-+\rangle = \langle +|\sigma_1^{(1)}|-\rangle\langle +|\sigma_2^{(2)}|+\rangle = b*e = be$$
$$\langle ++|(\sigma_1\sigma_2)^{(1)\otimes(2)}|--\rangle = \langle ++|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|--\rangle = \langle +|\sigma_1^{(1)}|-\rangle\langle +|\sigma_2^{(2)}|-\rangle = b*f = bf$$

Second row

$$\langle +-|(\sigma_1\sigma_2)^{(1)\otimes(2)}|++\rangle = \langle +-|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|++\rangle = \langle +|\sigma_1^{(1)}|+\rangle\langle -|\sigma_2^{(2)}|+\rangle = a*g = ag$$
$$\langle +-|(\sigma_1\sigma_2)^{(1)\otimes(2)}|+-\rangle = \langle +-|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|+-\rangle = \langle +|\sigma_1^{(1)}|+\rangle\langle -|\sigma_2^{(2)}|-\rangle = a*h = ah$$
$$\langle +-|(\sigma_1\sigma_2)^{(1)\otimes(2)}|-+\rangle = \langle +-|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|-+\rangle = \langle +|\sigma_1^{(1)}|-\rangle\langle -|\sigma_2^{(2)}|+\rangle = b*g = bg$$
$$\langle +-|(\sigma_1\sigma_2)^{(1)\otimes(2)}|--\rangle = \langle +-|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|--\rangle = \langle +|\sigma_1^{(1)}|-\rangle\langle -|\sigma_2^{(2)}|-\rangle = b*h = bh$$

Third row

$$\langle -+|(\sigma_1\sigma_2)^{(1)\otimes(2)}|++\rangle = \langle -+|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|++\rangle = \langle -|\sigma_1^{(1)}|+\rangle\langle +|\sigma_2^{(2)}|+\rangle = c*e = ce$$
$$\langle -+|(\sigma_1\sigma_2)^{(1)\otimes(2)}|+-\rangle = \langle -+|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|+-\rangle = \langle -|\sigma_1^{(1)}|+\rangle\langle +|\sigma_2^{(2)}|-\rangle = c*f = cf$$
$$\langle -+|(\sigma_1\sigma_2)^{(1)\otimes(2)}|-+\rangle = \langle -+|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|-+\rangle = \langle -|\sigma_1^{(1)}|-\rangle\langle +|\sigma_2^{(2)}|+\rangle = d*e = de$$
$$\langle -+|(\sigma_1\sigma_2)^{(1)\otimes(2)}|--\rangle = \langle -+|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|--\rangle = \langle -|\sigma_1^{(1)}|-\rangle\langle +|\sigma_2^{(2)}|-\rangle = d*f = df$$

Fourth row

$$\langle --|(\sigma_1\sigma_2)^{(1)\otimes(2)}|++\rangle = \langle --|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|++\rangle = \langle -|\sigma_1^{(1)}|+\rangle\langle -|\sigma_2^{(2)}|+\rangle = c*g = cg$$
$$\langle --|(\sigma_1\sigma_2)^{(1)\otimes(2)}|+-\rangle = \langle --|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|+-\rangle = \langle -|\sigma_1^{(1)}|+\rangle\langle -|\sigma_2^{(2)}|-\rangle = c*h = ch$$
$$\langle --|(\sigma_1\sigma_2)^{(1)\otimes(2)}|-+\rangle = \langle --|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|-+\rangle = \langle -|\sigma_1^{(1)}|-\rangle\langle -|\sigma_2^{(2)}|+\rangle = d*g = dg$$
$$\langle --|(\sigma_1\sigma_2)^{(1)\otimes(2)}|--\rangle = \langle --|\sigma_1^{(1)}\otimes\sigma_2^{(2)}|--\rangle = \langle -|\sigma_1^{(1)}|-\rangle\langle -|\sigma_2^{(2)}|-\rangle = d*h = dh$$

Putting the rows together gives

$$(\sigma_1\sigma_2)^{(1)\otimes(2)} = \sigma_1^{(1)\otimes(2)}\sigma_2^{(1)\otimes(2)} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

as above.

10.2.3 The Hamiltonian for the 3-D isotropic harmonic oscillator is

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2 + z^2\right)$$

We know the 1-D solutions:

$$H_1 = \frac{p_x^2}{2m} + \frac{m\omega^2}{2}x^2 \ \Rightarrow\ H_1|n\rangle = E_n|n\rangle \ \Rightarrow\ E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega$$

We can separate the 3-D case into 3 1-D cases:

$$\begin{aligned} H &= \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2 + z^2\right) \\ &= \left(\frac{p_x^2 +}{2m} + \frac{m\omega^2}{2}x^2\right) + \left(\frac{p_x^2 +}{2m} + \frac{m\omega^2}{2}x^2\right) + \left(\frac{p_x^2 +}{2m} + \frac{m\omega^2}{2}x^2\right) \\ &\equiv H_x + H_y + H_z \end{aligned}$$

The coordinates and momenta commute across dimensions, so the three 1-D Hamiltonians commute with each other. Thus we can find simultaneous eigenstates of the three 1-D Hamiltonians:

$$\begin{aligned} H_x|n_x\rangle &= E_{n_x}|n_x\rangle \ \Rightarrow\ E_{n_x} = \left(n_x + \tfrac{1}{2}\right)\hbar\omega \\ H_y|n_y\rangle &= E_{n_y}|n_y\rangle \ \Rightarrow\ E_{n_y} = \left(n_y + \tfrac{1}{2}\right)\hbar\omega \\ H_z|n_z\rangle &= E_{n_z}|n_z\rangle \ \Rightarrow\ E_{n_z} = \left(n_z + \tfrac{1}{2}\right)\hbar\omega \end{aligned}$$

The direct product states $|n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle = |n_x n_y n_z\rangle$ satisfy the 3-D eigenvalue equation:

$$H|E_n\rangle = E_n|E_n\rangle$$

$$\left(H_x + H_y + H_z\right)|E_n\rangle = E_n|E_n\rangle$$

$$\begin{aligned} \left(H_x + H_y + H_z\right)|n_x n_y n_z\rangle &= \left(H_x + H_y + H_z\right)|n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle \\ &= \left(H_x^{(x)} I^{(y)} I^{(z)} + I^{(x)} H_y^{(y)} I^{(z)} + I^{(x)} I^{(y)} H_z^{(z)}\right)|n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle \\ &= \left(E_{n_x} + E_{n_y} + E_{n_z}\right)|n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle \\ &= \left[\left(n_x + \tfrac{1}{2}\right)\hbar\omega + \left(n_y + \tfrac{1}{2}\right)\hbar\omega + \left(n_z + \tfrac{1}{2}\right)\hbar\omega\right]|n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle \\ &= \left(n_x + n_y + n_z + \tfrac{3}{2}\right)\hbar\omega|n_x n_y n_z\rangle \\ &= \left(n + \tfrac{3}{2}\right)\hbar\omega|n_x n_y n_z\rangle \end{aligned}$$

Hence, we conclude that

$$E_n = \left(n + \tfrac{3}{2}\right)\hbar\omega$$

where we define $n = n_x + n_y + n_z$, with each $n_i = 0, 1, 2, 3, \ldots$.

The single-dimension wave functions are (see p. 195)

$$\psi_n(x) = A_n e^{-\frac{m\omega}{2\hbar}x^2} H_n\left[x\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]$$

where H_n are the Hermite polynomials and A_n are the normalization constants. The 3-D wave functions are thus

$$\psi_{n_x,n_y,n_z}(x,y,z)=A_{n_x}A_{n_y}A_{n_z}e^{-\frac{m\omega}{2\hbar}x^2}H_{n_x}\left[x\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]e^{-\frac{m\omega}{2\hbar}y^2}H_{n_y}\left[y\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]e^{-\frac{m\omega}{2\hbar}z^2}H_{n_z}\left[z\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]$$

$$=A_{n_x}A_{n_y}A_{n_z}e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}H_{n_x}\left[x\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]H_{n_y}\left[y\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]H_{n_z}\left[z\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]$$

The parity of the Hermite polynomials can be seen by looking at the functions on page 195:

$$H_n(-x)=(-1)^n H_n(x)$$

i.e., they are alternately even and odd. The parity of the 3-D functions is thus

$$\psi_{n_x,n_y,n_z}(-x,-y,-z)=A_{n_x}A_{n_y}A_{n_z}e^{-\frac{m\omega}{2\hbar}\left((-x)^2+(-y)^2+(-z)^2\right)}H_{n_x}\left[-x\left(\tfrac{m\omega}{\hbar}\right)^{\frac{1}{2}}\right]H_{n_y}\left[-y\left(\tfrac{m\omega}{\hbar}\right)^{\frac{1}{2}}\right]H_{n_z}\left[-z\left(\tfrac{m\omega}{\hbar}\right)^{\frac{1}{2}}\right]$$

$$=A_{n_x}A_{n_y}A_{n_z}e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}H_{n_x}\left[-x\left(\tfrac{m\omega}{\hbar}\right)^{\frac{1}{2}}\right]H_{n_y}\left[-y\left(\tfrac{m\omega}{\hbar}\right)^{\frac{1}{2}}\right]H_{n_z}\left[-z\left(\tfrac{m\omega}{\hbar}\right)^{\frac{1}{2}}\right]$$

$$=A_{n_x}A_{n_y}A_{n_z}e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}(-1)^{n_x}H_{n_x}\left[x\left(\tfrac{m\omega}{\hbar}\right)^{\frac{1}{2}}\right](-1)^{n_y}H_{n_y}\left[y\left(\tfrac{m\omega}{\hbar}\right)^{\frac{1}{2}}\right](-1)^{n_z}H_{n_z}\left[z\left(\tfrac{m\omega}{\hbar}\right)^{\frac{1}{2}}\right]$$

$$=(-1)^{n_x+n_y+n_z}\psi_{n_x,n_y,n_z}(x,y,z)$$

Hence the parity is $(-1)^n$. The ground state is $|000\rangle$ and the wave function is

$$\psi_{000}(x,y,z)=A_0A_0A_0e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}H_0\left[x\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]H_0\left[y\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]H_0\left[z\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}e^{-\frac{m\omega}{2\hbar}r^2}$$

giving a simple form in spherical coordinates.

The first excited state is three-fold degenerate because there can be one quantum of excitation in any of the three dimensions. These three states are $|100\rangle,|010\rangle,|001\rangle$. The three wave functions are

$$\psi_{100}(x,y,z)=A_1A_0A_0e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}H_1\left[x\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]H_0\left[y\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]H_0\left[z\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\frac{1}{\sqrt{2}}e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}\left[2x\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\left(\frac{2m\omega}{\hbar}\right)^{1/2}e^{-\frac{m\omega}{2\hbar}r^2}\,r\sin\theta\cos\phi$$

$$\psi_{010}(x,y,z)=A_0A_1A_0e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}H_0\left[x\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]H_1\left[y\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]H_0\left[z\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\frac{1}{\sqrt{2}}e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}\left[2y\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\left(\frac{2m\omega}{\hbar}\right)^{1/2}e^{-\frac{m\omega}{2\hbar}r^2}r\sin\theta\sin\phi$$

$$\psi_{001}(x,y,z)=A_0A_0A_1e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}H_0\left[x\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]H_0\left[y\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]H_1\left[z\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\frac{1}{\sqrt{2}}e^{-\frac{m\omega}{2\hbar}\left(x^2+y^2+z^2\right)}\left[2z\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\left(\frac{2m\omega}{\hbar}\right)^{1/2}e^{-\frac{m\omega}{2\hbar}r^2}r\cos\theta$$

The degeneracy of a general state with energy $E_n=\left(n+\frac{3}{2}\right)\hbar\omega$ is obtained by counting the number of ways that three integers n_x, n_y, n_z (0 included) can be added to get the same result $n=n_x+n_y+n_z$. For a given energy state determined by n, there are n + 1 possible values for any one of the n_i; e.g., n_x could be 0 (with $n=n_y+n_z$) or n_x could be n (with $0=n_y+n_z$), and then all possibilities in between. For each value of n_x we then need to find out how many permutations there are for the other two indices. A table is helpful here:

n_x	n_yn_z	n_yn_z	n_yn_z	$\cdots$	n_yn_z	n_yn_z	# of states
n	00						1
$n-1$	10	01					2
$n-2$	20	11	02				3
$\vdots$	$\vdots$	$\vdots$	$\vdots$				$\vdots$
1	$n-1,0$	$n-2,1$	$n-3,2$	$\cdots$	$1,n-2$	$0,n-1$	n
0	$n0$	$n-1,1$	$n-2,2$	$\cdots$	$1,n-1$	$0n$	$n+1$

Note that this table does include all degenerate possibilities, e.g. n00, 0n0, 00n are all in the table as are (n-1)10, (n-1)01, 0(n-1)1, 1(n-1)0, 10(n-1), 01(n-1). Thus the total degeneracy is the sum of the integers in the last column (1 -> n+1):

$$Deg=\sum_{k=1}^{n+1}k=1+2+3+4+\ldots+(n-1)+n+(n+1)\quad\text{then group first and last terms etc to get}$$

$$=(n+2)+(n+2)+(n+2)+(n+2)+\ldots\qquad\tfrac{1}{2}(n+1)\ \text{times}$$

$$=\tfrac{1}{2}(n+1)(n+2)$$

10.3.1 Two identical bosons in states $|\phi\rangle$ and $|\psi\rangle$. Symmetrized state is

$$\begin{aligned}|\phi\psi,S\rangle &= \tfrac{1}{2}\left(1+P_{12}\right)|\phi\psi\rangle \\ &= \tfrac{1}{2}\left(|\phi\psi\rangle+|\psi\phi\rangle\right)\end{aligned}$$

This may not be normalized, so use

$$|\phi\psi,S\rangle = N\left(|\phi\psi\rangle+|\psi\phi\rangle\right)$$

and then find *N* by requiring normalization:

$$\begin{aligned}1 &= \langle\phi\psi,S|\phi\psi,S\rangle \\ &= \left(\langle\phi\psi|+\langle\psi\phi|\right)A^{*}A\left(|\phi\psi\rangle+|\psi\phi\rangle\right) \\ &= |N|^2\left(\langle\phi\psi|\phi\psi\rangle+\langle\phi\psi|\psi\phi\rangle+\langle\psi\phi|\phi\psi\rangle+\langle\psi\phi|\psi\phi\rangle\right) \\ &= |N|^2\left(\langle\phi|\phi\rangle\langle\psi|\psi\rangle+\langle\phi|\psi\rangle\langle\psi|\phi\rangle+\langle\psi|\phi\rangle\langle\phi|\psi\rangle+\langle\psi|\psi\rangle\langle\phi|\phi\rangle\right) \\ &= |N|^2\left(2\langle\phi|\phi\rangle\langle\psi|\psi\rangle+2\langle\phi|\psi\rangle\langle\psi|\phi\rangle\right) \\ &= 2|N|^2\left(1+|\langle\phi|\psi\rangle|^2\right)\end{aligned}$$

Choosing *N* to be real and positive gives

$$N = \frac{1}{\sqrt{2}\sqrt{1+|\langle\phi|\psi\rangle|^2}}$$

When the two states are orthogonal, we get the expected $1/\sqrt{2}$, but when they are not orthogonal we must include this overlap factor.

4. a) For two spin-½ particles, the possible two-particles spin states are $|++\rangle$, $|+-\rangle$, $|-+\rangle$ and $|--\rangle$. Symmetrizing these gives

$$\begin{aligned}S|++\rangle &= \tfrac{1}{2}\left(1+P_{12}\right)|++\rangle = \tfrac{1}{2}\left(|++\rangle+|++\rangle\right) = |++\rangle \\ S|+-\rangle &= \tfrac{1}{2}\left(1+P_{12}\right)|+-\rangle = \tfrac{1}{2}\left(|+-\rangle+|-+\rangle\right) \\ S|-+\rangle &= \tfrac{1}{2}\left(1+P_{12}\right)|-+\rangle = \tfrac{1}{2}\left(|-+\rangle+|+-\rangle\right) = \tfrac{1}{2}\left(|+-\rangle+|-+\rangle\right) \\ S|--\rangle &= \tfrac{1}{2}\left(1+P_{12}\right)|--\rangle = \tfrac{1}{2}\left(|--\rangle+|--\rangle\right) = |--\rangle\end{aligned}$$

Two of the states are the same, so we get 3 states. Normalizing (see above) gives the states

$$\begin{aligned}|++,S\rangle &= |++\rangle \\ |+-,S\rangle &= \tfrac{1}{\sqrt{2}}\left(|+-\rangle+|-+\rangle\right) \\ |--,S\rangle &= |--\rangle\end{aligned}$$

For the antisymmetric states, we get

$$\begin{aligned}A|++\rangle &= \tfrac{1}{2}\left(1-P_{12}\right)|++\rangle = \tfrac{1}{2}\left(|++\rangle-|++\rangle\right) = 0 \\ A|+-\rangle &= \tfrac{1}{2}\left(1-P_{12}\right)|+-\rangle = \tfrac{1}{2}\left(|+-\rangle-|-+\rangle\right) \\ A|-+\rangle &= \tfrac{1}{2}\left(1-P_{12}\right)|-+\rangle = \tfrac{1}{2}\left(|-+\rangle-|+-\rangle\right) = -\tfrac{1}{2}\left(|+-\rangle-|-+\rangle\right) \\ A|--\rangle &= \tfrac{1}{2}\left(1-P_{12}\right)|--\rangle = \tfrac{1}{2}\left(|--\rangle-|--\rangle\right) = 0\end{aligned}$$

Two of the states are null vectors, and two of the states differ by a sign so they are the same physical state, resulting in only 1 state. Normalizing gives the state

$$\left|+-,A\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|+-\right\rangle - \left|-+\right\rangle\right)$$

For spin 1, the possible two-particles states are $\left|11\right\rangle$, $\left|10\right\rangle$, $\left|1,-1\right\rangle$, $\left|01\right\rangle$, $\left|00\right\rangle$, $\left|0,-1\right\rangle$, $\left|-1,1\right\rangle$, $\left|-1,0\right\rangle$ and $\left|-1,-1\right\rangle$. Symmetrizing these gives

$$S\left|11\right\rangle = \tfrac{1}{2}\left(1+P_{12}\right)\left|11\right\rangle = \tfrac{1}{2}\left(\left|11\right\rangle + \left|11\right\rangle\right) = \left|11\right\rangle$$

$$S\left|10\right\rangle = \tfrac{1}{2}\left(1+P_{12}\right)\left|10\right\rangle = \tfrac{1}{2}\left(\left|10\right\rangle + \left|01\right\rangle\right)$$

$$S\left|1,-1\right\rangle = \tfrac{1}{2}\left(1+P_{12}\right)\left|1,-1\right\rangle = \tfrac{1}{2}\left(\left|1,-1\right\rangle + \left|-1,1\right\rangle\right)$$

$$S\left|01\right\rangle = \tfrac{1}{2}\left(1+P_{12}\right)\left|01\right\rangle = \tfrac{1}{2}\left(\left|01\right\rangle + \left|10\right\rangle\right) = \tfrac{1}{2}\left(\left|10\right\rangle + \left|01\right\rangle\right)$$

$$S\left|00\right\rangle = \tfrac{1}{2}\left(1+P_{12}\right)\left|00\right\rangle = \tfrac{1}{2}\left(\left|00\right\rangle + \left|00\right\rangle\right) = \left|00\right\rangle$$

$$S\left|0,-1\right\rangle = \tfrac{1}{2}\left(1+P_{12}\right)\left|0,-1\right\rangle = \tfrac{1}{2}\left(\left|0,-1\right\rangle + \left|-1,0\right\rangle\right)$$

$$S\left|-1,1\right\rangle = \tfrac{1}{2}\left(1+P_{12}\right)\left|-1,1\right\rangle = \tfrac{1}{2}\left(\left|-1,1\right\rangle + \left|1,-1\right\rangle\right) = \tfrac{1}{2}\left(\left|1,-1\right\rangle + \left|-1,1\right\rangle\right)$$

$$S\left|-1,0\right\rangle = \tfrac{1}{2}\left(1+P_{12}\right)\left|-1,0\right\rangle = \tfrac{1}{2}\left(\left|-1,0\right\rangle + \left|0,-1\right\rangle\right) = \tfrac{1}{2}\left(\left|0,-1\right\rangle + \left|-1,0\right\rangle\right)$$

$$S\left|-1,-1\right\rangle = \tfrac{1}{2}\left(1+P_{12}\right)\left|-1,-1\right\rangle = \tfrac{1}{2}\left(\left|-1,-1\right\rangle + \left|-1,-1\right\rangle\right) = \left|-1,-1\right\rangle$$

Of the 9 states, there are 3 pairs of identical states, so we get 6 states. Normalizing gives the states

$$\left|11,S\right\rangle = \left|11\right\rangle$$

$$\left|10,S\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|10\right\rangle + \left|01\right\rangle\right)$$

$$\left|1,-1,S\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|1,-1\right\rangle + \left|-1,1\right\rangle\right)$$

$$\left|00,S\right\rangle = \left|00\right\rangle$$

$$\left|0,-1,S\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|0,-1\right\rangle + \left|-1,0\right\rangle\right)$$

$$\left|-1,-1,S\right\rangle = \left|-1,-1\right\rangle$$

For the antisymmetric states, we get

$$A\left|11\right\rangle = \tfrac{1}{2}\left(1-P_{12}\right)\left|11\right\rangle = \tfrac{1}{2}\left(\left|11\right\rangle - \left|11\right\rangle\right) = 0$$

$$A\left|10\right\rangle = \tfrac{1}{2}\left(1-P_{12}\right)\left|10\right\rangle = \tfrac{1}{2}\left(\left|10\right\rangle - \left|01\right\rangle\right)$$

$$A\left|1,-1\right\rangle = \tfrac{1}{2}\left(1-P_{12}\right)\left|1,-1\right\rangle = \tfrac{1}{2}\left(\left|1,-1\right\rangle - \left|-1,1\right\rangle\right)$$

$$A\left|01\right\rangle = \tfrac{1}{2}\left(1-P_{12}\right)\left|01\right\rangle = \tfrac{1}{2}\left(\left|01\right\rangle - \left|10\right\rangle\right) = -\tfrac{1}{2}\left(\left|10\right\rangle - \left|01\right\rangle\right)$$

$$A\left|00\right\rangle = \tfrac{1}{2}\left(1-P_{12}\right)\left|00\right\rangle = \tfrac{1}{2}\left(\left|00\right\rangle - \left|00\right\rangle\right) = 0$$

$$A\left|0,-1\right\rangle = \tfrac{1}{2}\left(1-P_{12}\right)\left|0,-1\right\rangle = \tfrac{1}{2}\left(\left|0,-1\right\rangle - \left|-1,0\right\rangle\right)$$

$$A\left|-1,1\right\rangle = \tfrac{1}{2}\left(1-P_{12}\right)\left|-1,1\right\rangle = \tfrac{1}{2}\left(\left|-1,1\right\rangle - \left|1,-1\right\rangle\right) = -\tfrac{1}{2}\left(\left|1,-1\right\rangle - \left|-1,1\right\rangle\right)$$

$$A\left|-1,0\right\rangle = \tfrac{1}{2}\left(1-P_{12}\right)\left|-1,0\right\rangle = \tfrac{1}{2}\left(\left|-1,0\right\rangle - \left|0,-1\right\rangle\right) = -\tfrac{1}{2}\left(\left|0,-1\right\rangle - \left|-1,0\right\rangle\right)$$

$$A\left|-1,-1\right\rangle = \tfrac{1}{2}\left(1-P_{12}\right)\left|-1,-1\right\rangle = \tfrac{1}{2}\left(\left|-1,-1\right\rangle - \left|-1,-1\right\rangle\right) = 0$$

Three of the states are null vectors, and there are three pairs that differ by a sign so they are the same physical states, resulting in only 3 states. Normalizing gives the states

$$|10,A\rangle = \tfrac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)$$
$$|1,-1,A\rangle = \tfrac{1}{\sqrt{2}}\left(|1,-1\rangle - |-1,1\rangle\right)$$
$$|0,-1,A\rangle = \tfrac{1}{\sqrt{2}}\left(|0,-1\rangle - |-1,0\rangle\right)$$

b) For the spin-1/2 case, there are 4 states in the direct product Hilbert space ($|{+}{+}\rangle$, $|{+}{-}\rangle$, $|{-}{+}\rangle$ and $|{-}{-}\rangle$). The symmetric space has 3 states and the antisymmetric space has 1 state, so they collectively cover the direct product Hilbert space.

For the spin-1 case, there are 9 states in the direct product Hilbert space ($|11\rangle$, $|10\rangle$, $|1,-1\rangle$, $|01\rangle$, $|00\rangle$, $|0,-1\rangle$, $|-1,1\rangle$ $|-1,0\rangle$ and $|-1,-1\rangle$). The symmetric space has 6 states and the antisymmetric space has 3 states, so they collectively cover the direct product Hilbert space.

In both cases, the number of states in the symmetric space is greater than the number of states in the antisymmetric space: Spin ½: 3:1, Spin 1: 6:3.