1. a) To find the energy eigenvalues and eigenstates, diagonalize the Hamiltonian:

$$H \doteq E_0 \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$$

$$\begin{vmatrix} 6E_0 - \lambda & 2E_0 \\ 2E_0 & 9E_0 - \lambda \end{vmatrix} = 0$$

$$(6E_0 - \lambda)(9E_0 - \lambda) - 4E_0^2 = 0$$

$$\lambda^2 - 15E_0\lambda + 50E_0^2 = 0$$

$$(\lambda - 5E_0)(\lambda - 10E_0) = 0$$

$$\lambda = 5E_0 \ , 10E_0$$

$$H|E_i\rangle = E_i|E_i\rangle$$

$$E_0 \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = E_i \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$$

$$E_0(6\alpha_i + 2\beta_i) = E_i\alpha_i$$

$$E_1 = 5E_0 : \ E_0(6\alpha_1 + 2\beta_1) = 5E_0\alpha_1 \Rightarrow \alpha_1 = -2\beta_1$$

$$E_2 = 10E_0 : \ E_0(6\alpha_2 + 2\beta_2) = 10E_0\alpha_2 \Rightarrow \alpha_2 = \tfrac{1}{2}\beta_2$$

$$|E_1\rangle = \tfrac{1}{\sqrt{5}}\left(2|a_1\rangle - |a_2\rangle\right) \doteq \tfrac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$|E_2\rangle = \tfrac{1}{\sqrt{5}}\left(|a_1\rangle + 2|a_2\rangle\right) \doteq \tfrac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Let's now find the time dependent state vector using the propagator.

$$\begin{aligned}
|\psi(t)\rangle &= U(t)|\psi(0)\rangle \\
&= \sum_{\alpha}\sum_{E} |E,\alpha\rangle\langle E,\alpha| e^{-iEt/\hbar}|\psi(0)\rangle \\
&= \sum_{i} \langle E_i|\psi(0)\rangle e^{-iE_it/\hbar}|E_i\rangle \\
&= \sum_{i} \langle E_i|\tfrac{1}{5}\left[3|a_1\rangle + 4|a_2\rangle\right] e^{-iE_it/\hbar}|E_i\rangle \\
&= \tfrac{1}{5}\left[3\langle E_1|a_1\rangle + 4\langle E_1|a_2\rangle\right] e^{-iE_1t/\hbar}|E_1\rangle + \tfrac{1}{5}\left[3\langle E_2|a_1\rangle + 4\langle E_2|a_2\rangle\right] e^{-iE_2t/\hbar}|E_2\rangle \\
&= \tfrac{1}{5}\left[3\tfrac{2}{\sqrt{5}} - 4\tfrac{1}{\sqrt{5}}\right] e^{-iE_1t/\hbar}|E_1\rangle + \tfrac{1}{5}\left[3\tfrac{1}{\sqrt{5}} + 4\tfrac{2}{\sqrt{5}}\right] e^{-iE_2t/\hbar}|E_2\rangle \\
&= \tfrac{2}{5\sqrt{5}} e^{-iE_1t/\hbar}|E_1\rangle + \tfrac{11}{5\sqrt{5}} e^{-iE_2t/\hbar}|E_2\rangle
\end{aligned}$$

The possible energies that can be measured are the eigenvalues $E_1 = 5E_0$, $E_2 = 10E_0$. The probabilities are

$$\mathcal{P}_{E_n} = \left|\langle E_n|\psi(t)\rangle\right|^2$$

$$\mathcal{P}_{E_n} = \left|\langle E_n|\tfrac{1}{5\sqrt{5}}\left[2e^{-i\frac{5E_0}{\hbar}t}|E_1\rangle + 11e^{-i\frac{10E_0}{\hbar}t}|E_2\rangle\right]\right|^2$$

$$\boxed{\begin{aligned}\mathcal{P}_{5E_0} &= \left(\tfrac{2}{5\sqrt{5}}\right)^2 = \tfrac{4}{125} = 0.032\\ \mathcal{P}_{10E_0} &= \left(\tfrac{11}{5\sqrt{5}}\right)^2 = \tfrac{121}{125} = 0.968\end{aligned}}$$

Since the energy states are stationary states, this result is time independent.

b) Now find the expectation value of the operator A. The operator A has eigenvalues a_1 and a_2, so the matrix representation of A in the $|a_i\rangle$ basis is

$$A \doteq \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

To calculate the expectation value, write the state vector in the $|a_i\rangle$ basis

$$\begin{aligned}|\psi(t)\rangle &= \tfrac{1}{5\sqrt{5}}e^{-i\frac{5E_0}{\hbar}t}\left[2|E_1\rangle + 11e^{-i\frac{5E_0}{\hbar}t}|E_2\rangle\right]\\ &\doteq \tfrac{1}{5\sqrt{5}}e^{-i\frac{5E_0}{\hbar}t}\left\{\tfrac{2}{\sqrt{5}}\begin{pmatrix}2\\-1\end{pmatrix} + e^{-i\frac{5E_0}{\hbar}t}\,\tfrac{11}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}\right\}\end{aligned}$$

Hence

$$\begin{aligned}\langle A\rangle &= \langle\psi(t)|A|\psi(t)\rangle\\ &= \tfrac{1}{625}\left\{2\begin{pmatrix}2 & -1\end{pmatrix} + 11e^{+i\frac{5E_0}{\hbar}t}\begin{pmatrix}1 & 2\end{pmatrix}\right\}\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}\left\{2\begin{pmatrix}2\\-1\end{pmatrix} + 11e^{-i\frac{5E_0}{\hbar}t}\begin{pmatrix}1\\2\end{pmatrix}\right\}\\ &= \tfrac{1}{625}\left\{2\begin{pmatrix}2 & -1\end{pmatrix} + 11e^{+i\frac{5E_0}{\hbar}t}\begin{pmatrix}1 & 2\end{pmatrix}\right\}\left\{2\begin{pmatrix}2a_1\\-a_2\end{pmatrix} + 11e^{-i\frac{5E_0}{\hbar}t}\begin{pmatrix}a_1\\2a_2\end{pmatrix}\right\}\\ &= \tfrac{1}{625}\left\{4(4a_1 + a_2) + 22e^{+i\frac{5E_0}{\hbar}t}(2a_1 - 2a_2) + 22e^{-i\frac{5E_0}{\hbar}t}(2a_1 - 2a_2) + 121(a_1 + 4a_2)\right\}\\ &= \tfrac{1}{625}\left\{137a_1 + 488a_2 + 88(a_1 - a_2)\cos\left(\tfrac{5E_0}{\hbar}t\right)\right\}\end{aligned}$$

which oscillates at the Bohr frequency, as one might expect.

2. The initial momentum space wave function is

$$\psi(p,0) = A\int_{-\infty}^{\infty} e^{-|x|/x_0}\ e^{-ipx/\hbar}\ dx$$

a) To find the momentum space wave function, do the integral in its definition

$$\psi(p,0) = A\int_{-\infty}^{\infty} e^{-|x|/x_0}\ e^{-ipx/\hbar}\ dx = A\left[\int_{-\infty}^{0} e^{x/x_0}\ e^{-ipx/\hbar}\,dx\ + \int_{0}^{\infty} e^{-x/x_0}\ e^{-ipx/\hbar}\,dx\ \right]$$

$$= A\left[\int_{-\infty}^{0}\ e^{(1/x_0 - ip/\hbar)x}\,dx\ + \int_{0}^{\infty} e^{(-1/x_0 - ip/\hbar)x}\,dx\ \right]$$

$$= A\left\{\left[\frac{e^{(1/x_0 - ip/\hbar)x}}{(1/x_0 - ip/\hbar)}\right]_{-\infty}^{0} + \left[\frac{e^{(-1/x_0 - ip/\hbar)x}}{(-1/x_0 - ip/\hbar)}\right]_{0}^{\infty}\right\}$$

$$= A\left\{\frac{1}{(1/x_0 - ip/\hbar)} - \frac{1}{(-1/x_0 - ip/\hbar)}\right\}$$

$$= A\left\{\frac{(1/x_0 + ip/\hbar)}{(1/x_0^2 + p^2/\hbar^2)} + \frac{(1/x_0 - ip/\hbar)}{(1/x_0^2 + p^2/\hbar^2)}\right\}$$

$$= \frac{2A\hbar^2}{x_0\left(p^2 + \hbar^2/x_0^2\right)}$$

Get A by normalizing here or in part (b)

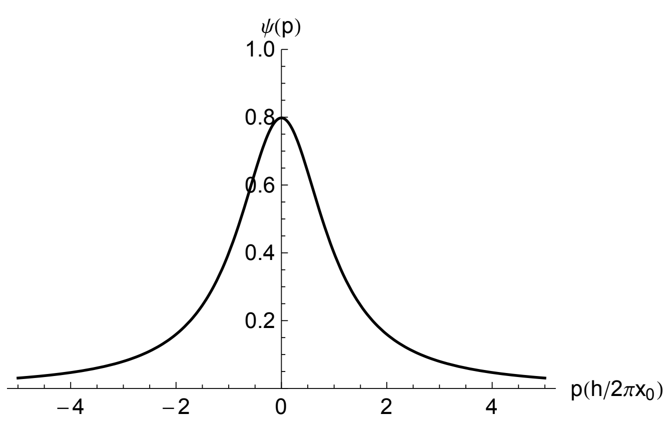
$$1 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty}\langle\psi|p\rangle\langle p|\psi\rangle dp = \int_{-\infty}^{\infty}|\psi(p)|^2\,dp = \int_{-\infty}^{\infty}\left|\frac{2A\hbar^2}{x_0\left(p^2 + \hbar^2/x_0^2\right)}\right|^2 dp$$

$$= \frac{4|A|^2\hbar^4}{x_0^2}2\int_0^{\infty}\frac{1}{\left(p^2 + \hbar^2/x_0^2\right)^2}dp = \frac{4|A|^2\hbar^4}{x_0^2}\frac{2\pi}{4\left(\hbar/x_0\right)^3} = |A|^2\,2\pi\hbar x_0$$

Choose phase real and positive ($A = 1/\sqrt{2\pi\hbar x_0}$), giving

$$\psi(p,0) = \sqrt{\frac{2\hbar^3}{\pi x_0^3}}\frac{1}{\left(p^2 + \hbar^2/x_0^2\right)}$$

Plot shown below



b) The probability of a position measurement is

$$\mathcal{P}(-a \le x \le a) = \int_{-a}^{a} \mathcal{P}(x)dx = \int_{-a}^{a} |\psi(x)|^2 dx$$

First find the position space wave function

$$\psi(x) = \langle x|\psi\rangle = \int_{-\infty}^{\infty} \langle x|p\rangle\langle p|\psi\rangle dp = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \psi(p)dp$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} A \int_{-\infty}^{\infty} e^{-|x'|/x_0} \ e^{-ipx'/\hbar} \ dx' \, dp$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-|x'|/x_0} \left\{ \int_{-\infty}^{\infty} e^{i(x-x')p/\hbar} \ dp \right\} dx'$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-|x'|/x_0} \left\{ 2\pi\hbar\delta(x-x') \right\} dx'$$

$$= A\sqrt{2\pi\hbar}\, e^{-|x|/x_0}$$

$$= \frac{1}{\sqrt{x_0}} e^{-|x|/x_0}$$

This result can also be obtained by noting the similarity between the initial state and the Fourier transform relation between wave functions in position space and momentum space representations:

$$\psi(p) = \langle p|\psi\rangle = \int_{-\infty}^{\infty} \langle p|x\rangle\langle x|\psi\rangle dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x)dx$$

$$\psi(p,0) = A\int_{-\infty}^{\infty} e^{-|x|/x_0} \ e^{-ipx/\hbar} \ dx \quad \Rightarrow \quad \psi(x) = A\sqrt{2\pi\hbar} \ e^{-|x|/x_0}$$

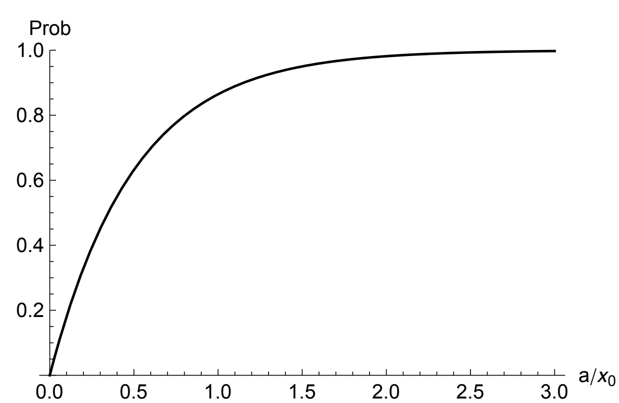
Plug into the probability and integrate:

$$\mathcal{P}(-a \le x \le a) = \int_{-a}^{a} \mathcal{P}(x)dx = \int_{-a}^{a} |\psi(x)|^2 dx$$

$$= \int_{-a}^{a} \left| \frac{1}{\sqrt{x_0}} e^{-|x|/x_0} \right|^2 dx = \frac{1}{x_0} \int_{-a}^{a} e^{-2|x|/x_0} dx = \frac{2}{x_0} \int_{0}^{a} e^{-2x/x_0} dx$$

$$= \frac{2}{x_0} \left[-\frac{x_0}{2} e^{-2x/x_0} \right]_0^a = 1 - e^{-2a/x_0}$$

sketched below.



c) To find the time-dependent momentum space wave function, apply the propagator, translating from the discrete energy case to the continuous energy case and then the momentum case:

$$U(t) = \sum_{\alpha} \sum_{E} |E,\alpha\rangle\langle E,\alpha| e^{-iEt/\hbar} \quad \Rightarrow \quad U(t) = \sum_{\alpha} \int_0^{\infty} |E,\alpha\rangle\langle E,\alpha| e^{-iEt/\hbar} \, dE$$

$$\Rightarrow \quad U(t) = \int_{-\infty}^{\infty} |p\rangle\langle p| e^{-ip^2t/2m\hbar} \, dp$$

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = \int_{-\infty}^{\infty} |p\rangle\langle p| e^{-ip^2t/2m\hbar} \, dp |\psi(0)\rangle$$

$$= \int_{-\infty}^{\infty} |p\rangle\langle p|\psi(0)\rangle e^{-ip^2t/2m\hbar} \, dp = \int_{-\infty}^{\infty} |p\rangle \psi(p,0) e^{-ip^2t/2m\hbar} \, dp$$

Don't waste time with the integral yet, just project into momentum space:

$$\psi(p,t) = \langle p|\psi(t)\rangle = \langle p| \int_{-\infty}^{\infty} |p'\rangle \psi(p',0) e^{-ip'^2t/2m\hbar} \, dp' = \int_{-\infty}^{\infty} \langle p|p'\rangle \psi(p',0) e^{-ip'^2t/2m\hbar} \, dp'$$

$$= \int_{-\infty}^{\infty} \delta(p-p') \psi(p',0) e^{-ip'^2t/2m\hbar} \, dp' = \psi(p,0) e^{-ip^2t/2m\hbar}$$

This just says that each momentum component evolves with the Schrödinger phase for that energy eigenstate (since momentum eigenstates are energy eigenstates). Hence, the time-dependent momentum space wave function is

$$\psi(p,t) = \sqrt{\frac{2\hbar^3}{\pi x_0^3}} \frac{1}{\left(p^2 + \hbar^2/x_0^2\right)} e^{-ip^2t/2m\hbar}$$

The probability density in momentum space is

$$\mathcal{P}(p,t) = \left|\langle p|\psi(t)\rangle\right|^2 = \left|\psi(p,t)\right|^2$$

Using the momentum space wave function from above gives

$$\mathcal{P}(p,t) = \left|\psi(p,t)\right|^2 = \left|\psi(p,0) e^{-ip^2t/2m\hbar}\right|^2 = \left|\psi(p,0)\right|^2$$

$$= \frac{2\hbar^3}{\pi x_0^3} \frac{1}{\left(p^2 + \hbar^2/x_0^2\right)^2}$$

This is time independent, which is what we expect for a free particle because there are no external forces. Probability density is shown below: