7.3.5 The expectation value of position is

$$\begin{aligned}\langle X\rangle &= \langle n|X|n\rangle \\ &= \int_{-\infty}^{\infty}\psi_n^*(x)\, x\, \psi_n(x)dx \\ &= \int_{-\infty}^{\infty} x|\psi_n(x)|^2\, dx\end{aligned}$$

The function x has odd spatial symmetry and the functions $|\psi_n(x)|^2$ have even spatial symmetry (including when $\psi_n(x)$ is odd), so the integral is zero.
The expectation value of momentum is

$$\begin{aligned}\langle P\rangle &= \langle n|P|n\rangle \\ &= \int_{-\infty}^{\infty}\psi_n^*(p)\, p\, \psi_n(p)dp \\ &= \int_{-\infty}^{\infty} p|\psi_n(p)|^2\, dp\end{aligned}$$

The function p has odd symmetry in momentum space and the functions $|\psi_n(p)|^2$ have even momentum space symmetry (including when $\psi_n(p)$ is odd), so the integral is zero.
Because these expectations values are zero, the uncertainties are simplified:

$$\Delta X = \sqrt{\left\langle \left(X-\langle X\rangle\right)^2\right\rangle} = \sqrt{\langle X^2\rangle - \langle X\rangle^2} = \sqrt{\langle X^2\rangle} \quad \text{since } \langle X\rangle = 0$$

$$\Delta P = \sqrt{\left\langle \left(P-\langle P\rangle\right)^2\right\rangle} = \sqrt{\langle P^2\rangle - \langle P\rangle^2} = \sqrt{\langle P^2\rangle} \quad \text{since } \langle P\rangle = 0$$

For the n = 1 state, the expectation values we need are (use the parametrization $\beta = \sqrt{m\omega/\hbar}$)

$$\begin{aligned}\langle X^2\rangle &= \int_{-\infty}^{\infty}\psi_1^*(x)x^2\psi_1(x)dx = \int_{-\infty}^{\infty} x^2|\psi_1(x)|^2\, dx \\ &= \int_{-\infty}^{\infty} x^2\left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}} 2\beta^2 x^2 e^{-\beta^2x^2}dx = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}} 2\beta^2\int_{-\infty}^{\infty} x^4 e^{-\beta^2x^2}\, dx \\ &= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}} 2\beta^2\frac{3\sqrt{\pi}}{4\beta^5} = \frac{3}{2\beta^2} = \frac{3\hbar}{2m\omega}\end{aligned}$$

and (here we use $\beta = \sqrt{1/\hbar m\omega}$)

$$\begin{aligned}\langle P^2\rangle &= \int_{-\infty}^{\infty}\psi_1^*(p)p^2\psi_1(p)dp = \int_{-\infty}^{\infty} p^2|\psi_1(p)|^2\, dp \\ &= \int_{-\infty}^{\infty} p^2\left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}} 2\beta^2 p^2 e^{-\beta^2p^2}dp = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}} 2\beta^2\int_{-\infty}^{\infty} p^4 e^{-\beta^2p^2}\, dx \\ &= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}} 2\beta^2\frac{3\sqrt{\pi}}{4\beta^5} = \frac{3}{2\beta^2} = \frac{3\hbar m\omega}{2}\end{aligned}$$

The uncertainty principle is $\Delta x \Delta p \geq \hbar/2$. For the $n = 1$ state we get:

$$\Delta p = \sqrt{\hbar m\omega\left(n+\tfrac{1}{2}\right)}$$

$$\begin{aligned}\Delta X \Delta P &= \sqrt{\langle X^2\rangle}\sqrt{\langle P^2\rangle}\\ &= \sqrt{\frac{3\hbar}{2m\omega}}\sqrt{\frac{3\hbar m\omega}{2}} = \frac{3}{2}\hbar\omega\end{aligned}$$

So the uncertainty relation is obeyed.

For the ground state we get:

$$\begin{aligned}\langle X^2\rangle &= \int_{-\infty}^{\infty}\psi_0^*(x)x^2\psi_0(x)dx = \int_{-\infty}^{\infty}x^2|\psi_0(x)|^2\,dx\\ &= \int_{-\infty}^{\infty}x^2\left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}}e^{-\beta^2x^2}dx = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}x^2e^{-\beta^2x^2}\,dx\\ &= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}}\frac{\sqrt{\pi}}{2\beta^3} = \frac{1}{2\beta^2} = \frac{\hbar}{2m\omega}\end{aligned}$$

and

$$\begin{aligned}\langle P^2\rangle &= \int_{-\infty}^{\infty}\psi_0^*(p)p^2\psi_0(p)dp = \int_{-\infty}^{\infty}p^2|\psi_0(p)|^2\,dp\\ &= \int_{-\infty}^{\infty}p^2\left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}}e^{-\beta^2p^2}dp = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}p^2e^{-\beta^2p^2}\,dx\\ &= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}}\frac{\sqrt{\pi}}{2\beta^3} = \frac{1}{2\beta^2} = \frac{\hbar m\omega}{2}\end{aligned}$$

yielding the uncertainty product:

$$\begin{aligned}\Delta X \Delta P &= \sqrt{\langle X^2\rangle}\sqrt{\langle P^2\rangle}\\ &= \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}} = \frac{1}{2}\hbar\omega\end{aligned}$$

which is the minimum uncertainty.

7.3.6 This new potential is "half" of the harmonic oscillator potential. Where the potentials are the same ($x > 0$), the solutions should be the same. But for the new potential, the wave functions must be zero for $x < 0$, where the potential energy is infinite. For the new wave functions to satisfy the continuity boundary condition, they must be zero at $x = 0$. The odd numbered "full" potential wave functions GO TO ZERO at $x = 0$ and so will work for this new potential (at least the part of them for $x < 0$). So the eigenstates of the new potential are the odd states of the "full" potential:

$$\psi_n(x) \; ; n = 1,3,5,7,\ldots$$

The energy eigenvalues are

$$E = \tfrac{3}{2}\hbar\omega, \tfrac{7}{2}\hbar\omega, \tfrac{11}{2}\hbar\omega, \tfrac{15}{2}\hbar\omega, \; .....$$

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega \qquad for \;\; n = 1,3,5,7,...$$

$$E_m = \left(2m + \tfrac{3}{2}\right)\hbar\omega \qquad for \;\; m = 0,1,2,3,...$$

7.4.1 The matrix elements of the ladder operators are given by

$$\langle m|a|n\rangle = \langle m|\sqrt{n}|n-1\rangle = \sqrt{n}\;\delta_{m,n-1} \qquad \langle m|a^\dagger|n\rangle = \langle m|\sqrt{n+1}|n+1\rangle = \sqrt{n+1}\;\delta_{m,n+1}$$

The matrix elements of X are

$$\begin{aligned}\langle m|X|n\rangle &= \langle m|\sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle m|\left(a^\dagger + a\right)|n\rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle m|a^\dagger|n\rangle + \langle m|a|n\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle m|\sqrt{n+1}|n+1\rangle + \langle m|\sqrt{n}|n-1\rangle\right] \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\;\delta_{m,n+1} + \sqrt{n}\;\delta_{m,n-1}\right]\end{aligned}$$

The matrix elements of P are

$$\begin{aligned}\langle m|P|n\rangle &= \langle m|\sqrt{\frac{\hbar m\omega}{2}}i\left(a^\dagger - a\right)|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle m|\left(a^\dagger - a\right)|n\rangle \\ &= i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle m|a^\dagger|n\rangle - \langle m|a|n\rangle\right] = i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle m|\sqrt{n+1}|n+1\rangle - \langle m|\sqrt{n}|n-1\rangle\right] \\ &= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\;\delta_{m,n+1} - \sqrt{n}\;\delta_{m,n-1}\right]\end{aligned}$$

Both agree with Exercise 7.3.4.

7.4.2 Calculate using the operators a and $a^\dagger$.

$$\langle X\rangle = \langle n|X|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a^\dagger + a|n\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|a^\dagger|n\rangle + \langle n|a|n\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|\sqrt{n+1}|n+1\rangle + \langle n|\sqrt{n}|n-1\rangle\right]$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\langle n|n+1\rangle + \sqrt{n}\langle n|n-1\rangle\right] = 0 \text{ since } \langle n|m\rangle = \delta_{nm}$$

$$\langle P\rangle = \langle n|P|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle n|a^\dagger - a|n\rangle$$
$$= i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle n|a^\dagger|n\rangle - \langle n|a|n\rangle\right] = i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle n|\sqrt{n+1}|n+1\rangle - \langle n|\sqrt{n}|n-1\rangle\right]$$
$$= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\langle n|n+1\rangle - \sqrt{n}\langle n|n-1\rangle\right] = 0 \text{ since } \langle n|m\rangle = \delta_{nm}$$

Note also that $\langle n|a^2|n\rangle = 0$ and $\langle n|\left(a^\dagger\right)^2|n\rangle = 0$ in a similar manner, so that

$$\langle X^2\rangle = \langle n|X^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|\left(a^\dagger + a\right)^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|\left(a^\dagger\right)^2 + a^\dagger a + aa^\dagger + a^2|n\rangle$$
$$= \frac{\hbar}{2m\omega}\langle n|a^\dagger a + aa^\dagger|n\rangle = \frac{\hbar}{2m\omega}\langle n|\sqrt{n}\sqrt{n} + \sqrt{n+1}\sqrt{n+1}|n\rangle$$
$$= \frac{\hbar}{2m\omega}(2n+1) = \frac{\hbar}{m\omega}\left(n+\tfrac{1}{2}\right)$$

$$\langle P^2\rangle = \langle n|P^2|n\rangle = -\frac{\hbar m\omega}{2}\langle n|\left(a^\dagger - a\right)^2|n\rangle = -\frac{\hbar m\omega}{2}\langle n|\left(a^\dagger\right)^2 - a^\dagger a - aa^\dagger + a^2|n\rangle$$
$$= \frac{\hbar m\omega}{2}\langle n|a^\dagger a + aa^\dagger|n\rangle = \frac{\hbar}{2m\omega}\langle n|\sqrt{n}\sqrt{n} + \sqrt{n+1}\sqrt{n+1}|n\rangle$$
$$= \frac{\hbar m\omega}{2}(2n+1) = \hbar m\omega\left(n+\tfrac{1}{2}\right)$$

The uncertainty principle is $\Delta X\Delta P \geq \hbar/2$ where

$$\Delta X = \sqrt{\left\langle\left(X - \langle X\rangle\right)^2\right\rangle} = \sqrt{\langle X^2\rangle - \langle X\rangle^2} = \sqrt{\langle X^2\rangle} \quad \text{since } \langle X\rangle = 0$$

$$\Delta P = \sqrt{\left\langle\left(P - \langle P\rangle\right)^2\right\rangle} = \sqrt{\langle P^2\rangle - \langle P\rangle^2} = \sqrt{\langle P^2\rangle} \quad \text{since } \langle P\rangle = 0$$

$$\Delta X = \sqrt{\frac{\hbar}{m\omega}\left(n+\tfrac{1}{2}\right)}$$

$$\Delta P = \sqrt{\hbar m\omega\left(n+\tfrac{1}{2}\right)}$$

$$\Delta X \Delta P = \sqrt{\frac{\hbar}{m\omega}\left(n+\tfrac{1}{2}\right)}\sqrt{\hbar m\omega\left(n+\tfrac{1}{2}\right)} = \left(n+\tfrac{1}{2}\right)\hbar \geq \frac{\hbar}{2}$$

So the uncertainty relation is obeyed.

7.4.5 The initial state is

$$\left|\psi(t=0)\right\rangle = \tfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

1) Time evolution:

$$\begin{aligned}\left|\psi(t)\right\rangle &= \tfrac{1}{\sqrt{2}}\left(e^{-iE_0 t/\hbar}|0\rangle + e^{-iE_1 t/\hbar}|1\rangle\right)\\ &= e^{-i\omega t/2}\,\tfrac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right)\end{aligned}$$

2) Expectation values:

$$\begin{aligned}\left\langle X(t)\right\rangle &= \left\langle \psi(t)\right|X\left|\psi(t)\right\rangle = \sqrt{\tfrac{\hbar}{2m\omega}}\left\langle \psi(t)\right|a^\dagger + a\left|\psi(t)\right\rangle\\ &= \sqrt{\tfrac{\hbar}{2m\omega}}e^{+i\omega t/2}\,\tfrac{1}{\sqrt{2}}\left(\langle 0| + e^{+i\omega t}\langle 1|\right)\left(a^\dagger + a\right)e^{-i\omega t/2}\,\tfrac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right)\\ &= \sqrt{\tfrac{\hbar}{2m\omega}}\,\tfrac{1}{2}\left[e^{-i\omega t}\langle 0|a|1\rangle + e^{+i\omega t}\langle 1|a^\dagger|0\rangle\right]\\ &= \sqrt{\tfrac{\hbar}{2m\omega}}\,\tfrac{1}{2}\left[e^{-i\omega t}\sqrt{1} + e^{+i\omega t}\sqrt{1}\right] = \sqrt{\tfrac{\hbar}{2m\omega}}\cos\omega t\\ &\Rightarrow \left\langle X(0)\right\rangle = \sqrt{\tfrac{\hbar}{2m\omega}}\end{aligned}$$

Momentum expectation value:

$$\begin{aligned}\left\langle P(t)\right\rangle &= \left\langle \psi(t)\right|P\left|\psi(t)\right\rangle = i\sqrt{\tfrac{m\omega\hbar}{2}}\left\langle \psi(t)\right|a^\dagger - a\left|\psi(t)\right\rangle\\ &= i\sqrt{\tfrac{m\omega\hbar}{2}}e^{+i\omega t/2}\,\tfrac{1}{\sqrt{2}}\left(\langle 0| + e^{+i\omega t}\langle 1|\right)\left(a^\dagger - a\right)e^{-i\omega t/2}\,\tfrac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right)\\ &= i\sqrt{\tfrac{m\omega\hbar}{2}}\,\tfrac{1}{2}\left[-e^{-i\omega t}\langle 0|a|1\rangle + e^{+i\omega t}\langle 1|a^\dagger|0\rangle\right]\\ &= i\sqrt{\tfrac{m\omega\hbar}{2}}\,\tfrac{1}{2}\left[-e^{-i\omega t}\sqrt{1} + e^{+i\omega t}\sqrt{1}\right] = -\sqrt{\tfrac{m\omega\hbar}{2}}\sin\omega t\\ &\Rightarrow \left\langle P(0)\right\rangle = 0\end{aligned}$$

3) Ehrenfest's theorem is

$$\begin{aligned}\frac{d}{dt}\langle X\rangle &= -\frac{i}{\hbar}\langle [X,H]\rangle\\ \frac{d}{dt}\langle P\rangle &= -\frac{i}{\hbar}\langle [P,H]\rangle\end{aligned}$$

For the harmonic oscillator, the commutators are

$$\begin{aligned}
[X,H] &= \left[\sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right), \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)\right] \\
&= \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left[\left(a^\dagger + a\right), a^\dagger a\right] = \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left\{\left[a^\dagger, a^\dagger a\right] + \left[a, a^\dagger a\right]\right\} \\
&= \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left\{a^\dagger a^\dagger a - a^\dagger a a^\dagger + a a^\dagger a - a^\dagger a a\right\} \\
&= \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left\{a^\dagger a^\dagger a - a^\dagger\left(a^\dagger a + 1\right) + \left(a^\dagger a + 1\right)a - a^\dagger a a\right\} \\
&= \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left\{a - a^\dagger\right\} \\
&= i\frac{\hbar}{m}P
\end{aligned}$$

and

$$\begin{aligned}
[P,H] &= \left[i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right), \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)\right] \\
&= i\hbar\omega\sqrt{\frac{\hbar m\omega}{2}}\left[\left(a^\dagger - a\right), a^\dagger a\right] = i\hbar\omega\sqrt{\frac{\hbar m\omega}{2}}\left\{\left[a^\dagger, a^\dagger a\right] - \left[a, a^\dagger a\right]\right\} \\
&= i\hbar\omega\sqrt{\frac{\hbar m\omega}{2}}\left\{a^\dagger a^\dagger a - a^\dagger a a^\dagger - a a^\dagger a + a^\dagger a a\right\} \\
&= i\hbar\omega\sqrt{\frac{\hbar m\omega}{2}}\left\{a^\dagger a^\dagger a - a^\dagger\left(a^\dagger a + 1\right) - \left(a^\dagger a + 1\right)a + a^\dagger a a\right\} \\
&= -i\hbar\omega\sqrt{\frac{\hbar m\omega}{2}}\left\{a + a^\dagger\right\} \\
&= -i\hbar m\omega^2 X
\end{aligned}$$

These give

$$\begin{aligned}
\frac{d}{dt}\langle X\rangle &= -\frac{i}{\hbar}\left\langle i\frac{\hbar}{m}P\right\rangle = \frac{\langle P\rangle}{m} \\
\frac{d}{dt}\langle P\rangle &= -\frac{i}{\hbar}\left\langle -i\hbar m\omega^2 X\right\rangle = -m\omega^2\langle X\rangle
\end{aligned}$$

Plug one equation into the other to get

$$\frac{d^2}{dt^2}\langle X\rangle = -\omega^2\langle X\rangle$$

The solution to this differential equation is

$$\langle X(t)\rangle = A\cos\omega t + B\sin\omega t$$

which also gives

$$\langle P(t)\rangle = m\langle \dot{X}(t)\rangle = -m\omega A\sin\omega t + m\omega B\cos\omega t$$

Using the $t = 0$ expectation values from (2) requires $B = 0$ and $A = \sqrt{\frac{\hbar}{2m\omega}}$. Hence we get

$$\langle X(t)\rangle = \sqrt{\tfrac{\hbar}{2m\omega}}\cos\omega t$$
$$\langle P(t)\rangle = -\sqrt{\tfrac{m\omega\hbar}{2}}\sin\omega t$$

which agree with the results from (2).

7.4.6 The expectation values we need are

$$\langle a(t)\rangle = \langle\psi(t)|a|\psi(t)\rangle$$
$$\langle a^\dagger(t)\rangle = \langle\psi(t)|a^\dagger|\psi(t)\rangle$$

A generic time-dependent wave function is

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}|n\rangle = e^{-i\omega t/2}\sum_{n=0}^{\infty} c_n e^{-in\omega t}|n\rangle$$

The expectation value of a is

$$\begin{aligned}
\langle a(t)\rangle &= \langle\psi(t)|a|\psi(t)\rangle \\
&= e^{+i\omega t/2}\sum_{m=0}^{\infty} c_m^* e^{+im\omega t}\langle m|\ a\ e^{-i\omega t/2}\sum_{n=0}^{\infty} c_n e^{-in\omega t}|n\rangle \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m^* c_n e^{+i(m-n)\omega t}\langle m|a|n\rangle \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m^* c_n e^{+i(m-n)\omega t}\langle m|\sqrt{n}\,|n-1\rangle \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m^* c_n e^{+i(m-n)\omega t}\sqrt{n}\,\delta_{m,n-1} \\
&= e^{-i\omega t}\sum_{m=0}^{\infty} c_m^* c_{m+1}\sqrt{m+1}
\end{aligned}$$

For $t = 0$, we get

$$\langle a(0)\rangle = \sum_{m=0}^{\infty} c_m^* c_{m+1}\sqrt{m+1}$$

Hence we get

$$\langle a(t)\rangle = e^{-i\omega t}\langle a(0)\rangle$$

Likewise

$$\begin{aligned}
\langle a^\dagger(t)\rangle &= \langle\psi(t)|a^\dagger|\psi(t)\rangle \\
&= e^{+i\omega t/2}\sum_{m=0}^{\infty} c_m^* e^{+im\omega t}\langle m|\;\; a^\dagger \;\; e^{-i\omega t/2}\sum_{n=0}^{\infty} c_n e^{-in\omega t}|n\rangle \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m^* c_n e^{+i(m-n)\omega t}\langle m|a^\dagger|n\rangle \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m^* c_n e^{+i(m-n)\omega t}\langle m|\sqrt{n+1}\,|n+1\rangle \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m^* c_n e^{+i(m-n)\omega t}\sqrt{n+1}\,\delta_{m,n+1} \\
&= e^{+i\omega t}\sum_{m=1}^{\infty} c_m^* c_{m-1}\sqrt{m} \\
&= e^{+i\omega t}\langle a^\dagger(0)\rangle
\end{aligned}$$