5.2.1 The particle is initially in the ground state of a box of size L. Hence the initial state is

$$\psi_{initial}(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right)$$

between $-L/2$ and $L/2$ and zero elsewhere. If the box is expanded "suddenly," this means the wave function remains unaltered. Now we want the probability that the particle in the expanded box is in the ground state of the expanded box. The probability that the initial state $|\psi_{initial}\rangle \doteq \psi_{initial}(x)$ is measured to be in the final state $|\psi_{final}\rangle \doteq \psi_{final}(x)$ is the square of the projection of the two states:

$$\begin{aligned}\mathcal{P}_{i\to f} &= \left|\langle\psi_{final}|\psi_{initial}\rangle\right|^2 \\ &= \left|\int_{-\infty}^{\infty}\psi^*_{final}(x)\psi_{initial}(x)dx\right|^2\end{aligned}$$

The ground state energy eigenfunction of the <u>expanded</u> box is (just change L to $2L$):

$$\varphi_{n=1\,for\,2L}(x) = \sqrt{\frac{2}{2L}}\cos\left(\frac{\pi x}{2L}\right)$$

between $-L$ and L.

The probability integral can be limited to the range between $-L/2$ and $L/2$ because the initial wave function is zero outside of that range. Hence

$$\begin{aligned}\mathcal{P}_{i\to n=1} &= \left|\int_{-\infty}^{\infty}\varphi_{n=1\,for\,2L}{}^*(x)\psi_{initial}(x)dx\right|^2 \\ &= \left|\int_{-L/2}^{L/2}\sqrt{\frac{2}{2L}}\cos\left(\frac{\pi x}{2L}\right)\sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right)dx\right|^2 \\ &= \left|\frac{2}{\sqrt{2}L}\left[\frac{L}{\pi}\sin\left(\frac{\pi x}{2L}\right)+\frac{L}{3\pi}\sin\left(\frac{3\pi x}{2L}\right)\right]_{-L/2}^{L/2}\right|^2 \\ &= \left|\frac{8}{3\pi}\right|^2 = \frac{64}{9\pi^2} \cong 0.72\end{aligned}$$

There is a 72% chance that the particle is measured in the ground state of the expanded box.

5.2.3 For a delta function potential, we must reconsider the continuity equation for the derivative of the wave function. The position representation of the energy eigenvalue equation is

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\varphi_E(x)=E\varphi_E(x)$$

Integrate from $-\varepsilon$ to ε to get

$$\int_{-\varepsilon}^{\varepsilon}-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi_E(x)dx+\int_{-\varepsilon}^{\varepsilon}V(x)\varphi_E(x)dx=E\int_{-\varepsilon}^{\varepsilon}\varphi_E(x)dx$$

In the limit that $\varepsilon\to 0$, the integral on the right is zero because the wave function must be finite. The first integral on the left yields the wave function first derivative, leaving

$$\left.\frac{d\varphi_E(x)}{dx}\right|_{\varepsilon}-\left.\frac{d\varphi_E(x)}{dx}\right|_{-\varepsilon}=\frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}V(x)\varphi_E(x)dx$$

If the potential energy is finite, then the integral on the right is zero, and the wave function derivative must be continuous. If the potential energy is infinite at the boundary, then the wave function derivative need not be continuous, as in the infinite square well problem and this delta function potential. We will use this below.
Start by solving the energy eigenvalue equation:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}-aV_0\delta(x)\right)\varphi_E(x)=E\varphi_E(x)$$

Outside of the potential well, $V=0$, so the solutions are real exponentials. Assume a solution of form

$$\varphi_E(x)=\begin{cases} Ae^{qx}+Be^{-qx}, & x<0 \\ Ce^{qx}+De^{-qx}, & x>0 \end{cases}$$

with $q=\sqrt{-2mE/\hbar^2}$. The boundary condition at infinity (normalizability) requires that $B=C=0$. The boundary condition on the continuity of the wave function at $x=0$ gives $A=D$, so

$$\varphi_E(x)=\begin{cases} Ae^{qx}, & x<0 \\ Ae^{-qx}, & x>0 \end{cases}$$

The boundary condition on the wave function derivative from above gives

$$\left.\frac{d\varphi_E(x)}{dx}\right|_{\varepsilon}-\left.\frac{d\varphi_E(x)}{dx}\right|_{-\varepsilon}=\frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}V(x)\varphi_E(x)dx$$

$$\lim_{\varepsilon\to 0}\left(-qAe^{-q\varepsilon}-qAe^{q\varepsilon}\right)=-aV_0\frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\delta(x)\varphi_E(x)dx$$

$$-2qA=-aV_0\frac{2m}{\hbar^2}\varphi_E(0)$$

$$-2qA=-aV_0\frac{2m}{\hbar^2}A$$

with the result

$$q=\frac{maV_0}{\hbar^2}$$

This yields the allowed energy

$$E=-\frac{\hbar^2q^2}{2m}=-\frac{ma^2V_0^2}{2\hbar^2}$$

Finally, we normalize the wave function:

$$1=\langle E|E\rangle=\int_{-\infty}^{\infty}\left|\varphi_E(x)\right|^2dx=2|A|^2\int_0^{\infty}e^{-2qx}\,dx=\frac{|A|^2}{q}$$

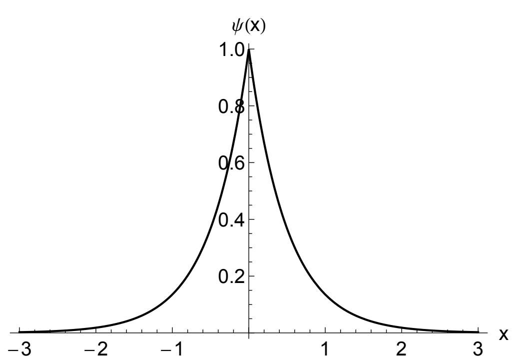
$$\Rightarrow\ A=\sqrt{q}=\frac{\sqrt{maV_0}}{\hbar}$$

The delta function potential energy well has only one bound state with solution

$$\varphi_E(x)=\frac{\sqrt{maV_0}}{\hbar}e^{-maV_0|x|/\hbar^2}\quad;\quad E=-\frac{ma^2V_0^2}{2\hbar^2}$$

There are no other bound state solutions. Plot below shows the cusp in the wave function with a change in slope of

$$\left.\frac{d\varphi_E(x)}{dx}\right|_{\varepsilon}-\left.\frac{d\varphi_E(x)}{dx}\right|_{-\varepsilon}=-2qA=-2\frac{(maV_0)^{3/2}}{\hbar^3}$$



3. a) Possible values of H must be eigenvalues. H is already diagonal, so the eigenvalues can be read off by inspection:

$$E = 1\hbar\omega_0,\, 2\hbar\omega_0\,(x2)\quad \text{(note degeneracy)}$$

Initial state is

$$\left|\psi(0)\right\rangle = \tfrac{1}{\sqrt{2}}\left|1\right\rangle + \tfrac{1}{2}\left|2\right\rangle + \tfrac{1}{2}\left|3\right\rangle$$

The probabilities are

$$\mathcal{P}_{E_1} = \left|\left\langle E_1|\psi\right\rangle\right|^2 = \left|\left(\left\langle 1\right|\right)\left(\tfrac{1}{\sqrt{2}}\left|1\right\rangle + \tfrac{1}{2}\left|2\right\rangle + \tfrac{1}{2}\left|3\right\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{E_2} = \left|\left\langle E_2|\psi\right\rangle\right|^2 = \sum_\alpha \left|\left\langle E_2,\alpha|\psi\right\rangle\right|^2 = \left|\left(\left\langle 2\right|\right)\left(\tfrac{1}{\sqrt{2}}\left|1\right\rangle + \tfrac{1}{2}\left|2\right\rangle + \tfrac{1}{2}\left|3\right\rangle\right)\right|^2 + \left|\left(\left\langle 3\right|\right)\left(\tfrac{1}{\sqrt{2}}\left|1\right\rangle + \tfrac{1}{2}\left|2\right\rangle + \tfrac{1}{2}\left|3\right\rangle\right)\right|^2$$

$$= \left|\tfrac{1}{2}\right|^2 + \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{2}$$

The two probabilities add to unity, as they must. The expectation values are

$$\left\langle H\right\rangle = \left\langle E\right\rangle = \sum_i E_i \mathcal{P}_{E_i} = 1\hbar\omega_0\tfrac{1}{2} + 2\hbar\omega_0\tfrac{1}{2} = \tfrac{3}{2}\hbar\omega_0$$

$$\left\langle E^2\right\rangle = \sum_i E_i^2 \mathcal{P}_{E_i} = \left(1\hbar\omega_0\right)^2\tfrac{1}{2} + \left(2\hbar\omega_0\right)^2\tfrac{1}{2} = \tfrac{5}{2}\hbar\omega_0$$

The uncertainty is

$$\Delta E = \sqrt{\left\langle E^2\right\rangle - \left\langle E\right\rangle^2} = \sqrt{\tfrac{5}{2}\left(\hbar\omega_0\right)^2 - \left(\tfrac{3}{2}\hbar\omega_0\right)^2} = \tfrac{1}{2}\hbar\omega_0$$

b) Find the wave function at time t by adding in the Schrödinger time-evolution phases:

$$\begin{aligned}\left|\psi(t)\right\rangle &= \tfrac{1}{\sqrt{2}}e^{-iE_1t/\hbar}\left|1\right\rangle + \tfrac{1}{2}e^{-iE_2t/\hbar}\left|2\right\rangle + \tfrac{1}{2}e^{-iE_2t/\hbar}\left|3\right\rangle \\ &= \tfrac{1}{\sqrt{2}}e^{-i\omega_0t}\left|1\right\rangle + \tfrac{1}{2}e^{-i2\omega_0t}\left|2\right\rangle + \tfrac{1}{2}e^{-i2\omega_0t}\left|3\right\rangle\end{aligned}$$

c) The expectation values of the operators A and B are:

$$\left\langle A\right\rangle = \left\langle\psi\right|A\left|\psi\right\rangle = \begin{pmatrix} \tfrac{1}{\sqrt{2}}e^{i\omega_0t} & \tfrac{1}{2}e^{i2\omega_0t} & \tfrac{1}{2}e^{i2\omega_0t}\end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0\end{pmatrix}\begin{pmatrix} \tfrac{1}{\sqrt{2}}e^{-i\omega_0t} \\ \tfrac{1}{2}e^{-i2\omega_0t} \\ \tfrac{1}{2}e^{-i2\omega_0t}\end{pmatrix}$$

$$= \begin{pmatrix} \tfrac{1}{\sqrt{2}}e^{i\omega_0t} & \tfrac{1}{2}e^{i2\omega_0t} & \tfrac{1}{2}e^{i2\omega_0t}\end{pmatrix}\begin{pmatrix} a\tfrac{1}{\sqrt{2}}e^{-i\omega_0t} \\ a\tfrac{1}{2}e^{-i2\omega_0t} \\ a\tfrac{1}{2}e^{-i2\omega_0t}\end{pmatrix} = \tfrac{1}{2}a + \tfrac{1}{4}a + \tfrac{1}{4}a = a$$

and

$$\langle B\rangle = \langle\psi|B|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{i\omega_0 t} & \frac{1}{2}e^{i2\omega_0 t} & \frac{1}{2}e^{i2\omega_0 t}\end{pmatrix}\begin{pmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & b\end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}}e^{-i\omega_0 t} \\ \frac{1}{2}e^{-i2\omega_0 t} \\ \frac{1}{2}e^{-i2\omega_0 t}\end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}e^{i\omega_0 t} & \frac{1}{2}e^{i2\omega_0 t} & \frac{1}{2}e^{i2\omega_0 t}\end{pmatrix}\begin{pmatrix} b\frac{1}{2}e^{-i2\omega_0 t} \\ b\frac{1}{\sqrt{2}}e^{-i\omega_0 t} \\ b\frac{1}{2}e^{-i2\omega_0 t}\end{pmatrix} = \frac{1}{2\sqrt{2}}be^{-i\omega_0 t} + \frac{1}{2\sqrt{2}}be^{+i\omega_0 t} + \frac{1}{4}b$$

$$= b\left(\tfrac{1}{4} + \tfrac{1}{\sqrt{2}}\cos\omega_0 t\right)$$

Note that the expectation value of A is time independent, while the expectation value of B is time dependent. This is related to whether or not the operators commute with the Hamiltonian H. A commutes with H, but B does not:

$$[H,A] = HA - AH \doteq \hbar\omega_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{pmatrix}a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix} - a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}\hbar\omega_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{pmatrix}$$

$$\doteq a\hbar\omega_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0\end{pmatrix} - a\hbar\omega_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0\end{pmatrix} = 0$$

and

$$[H,B] = HB - BH \doteq \hbar\omega_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{pmatrix}b\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix} - b\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}\hbar\omega_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{pmatrix}$$

$$\doteq b\hbar\omega_0\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2\end{pmatrix} - b\hbar\omega_0\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2\end{pmatrix} \doteq b\hbar\omega_0\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} \neq 0$$

4. The initial wave function is

$$\psi(x,0) = A\int_{-\infty}^{\infty} e^{-|p|/p_0}\, e^{ipx/\hbar}\, dp$$

a) The probability of a momentum measurement is

$$\mathcal{P}(-p_1 \le p \le p_1) = \int_{-p_1}^{p_1}\mathcal{P}(p)dp = \int_{-p_1}^{p_1}|\psi(p)|^2\, dp$$

Now find the momentum space wave function

$$\psi(p) = \langle p|\psi\rangle = \int_{-\infty}^{\infty} \langle p|x\rangle\langle x|\psi\rangle dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi(x)dx$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} A \int_{-\infty}^{\infty} e^{-|p'|/p_0}\ e^{ip'x/\hbar}\ dp'\,dx$$
$$= \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-|p'|/p_0} \left\{ \int_{-\infty}^{\infty} e^{i(p'-p)x/\hbar}\ dx \right\} dp'$$
$$= \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-|p'|/p_0} \left\{ 2\pi\hbar\delta(p'-p) \right\} dp'$$
$$= A\sqrt{2\pi\hbar}\, e^{-|p|/p_0}$$

This result can also be obtained by noting the similarity between the initial state and the Fourier transform relation between wave functions in position space and momentum space representations:

$$\psi(x) = \langle x|\psi\rangle = \int_{-\infty}^{\infty} \langle x|p\rangle\langle p|\psi\rangle dp = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\psi(p)dp$$

$$\psi(x,0) = A\int_{-\infty}^{\infty} e^{-|p|/p_0}\ e^{ipx/\hbar}\ dp \ \Rightarrow\ \psi(p) = A\sqrt{2\pi\hbar}\ e^{-|p|/p_0}$$

Plug into the probability:

$$\mathcal{P}(-p_1 \le p \le p_1) = \int_{-p_1}^{p_1} |\psi(p)|^2\, dp = \int_{-p_1}^{p_1} \left| A\sqrt{2\pi\hbar}\, e^{-|p|/p_0} \right|^2 dp$$
$$= |A|^2\, 2\pi\hbar \int_{-p_1}^{p_1} e^{-2|p|/p_0}\, dp = |A|^2\, 2\pi\hbar 2 \int_0^{p_1} e^{-2p/p_0}\, dp$$
$$= |A|^2\, 4\pi\hbar \left[-\frac{p_0}{2} e^{-2p/p_0} \right]_0^{p_1} = |A|^2\, 2\pi\hbar p_0 \left(1 - e^{-2p_1/p_0}\right)$$

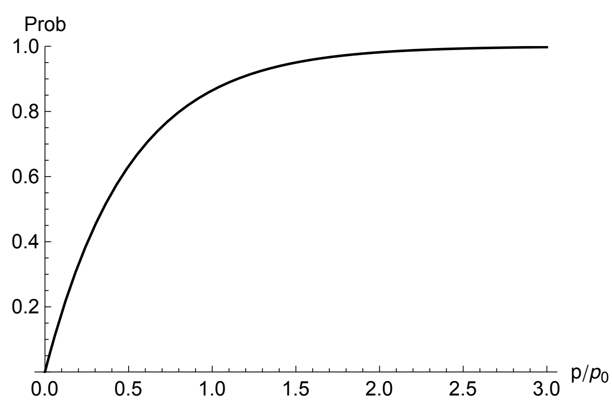
Note that the probability with $p_1 = \infty$ must be unity, so the normalization factor A must be

$$|A|^2 = \frac{1}{2\pi\hbar p_0} \Rightarrow A = \frac{1}{\sqrt{2\pi\hbar p_0}}$$

where we have chosen the real, positive solution as a choice of overall phase. This gives

$$\mathcal{P}(-p_1 \le p \le p_1) = \left(1 - e^{-2p_1/p_0}\right)$$

sketched below.



b) To find the wave function, do the integral in its definition

$$\psi(x,0)=\frac{1}{\sqrt{2\pi\hbar p_0}}\int_{-\infty}^{\infty}e^{-|p|/p_0}\ e^{ipx/\hbar}\ dp=\frac{1}{\sqrt{2\pi\hbar p_0}}\left[\int_{-\infty}^{0}e^{p/p_0}\ e^{ipx/\hbar}\ dp+\int_{0}^{\infty}e^{-p/p_0}\ e^{ipx/\hbar}\ dp\right]$$

$$=\frac{1}{\sqrt{2\pi\hbar p_0}}\left[\int_{-\infty}^{0}\ e^{(ix/\hbar+1/p_0)p}\ dp+\int_{0}^{\infty}e^{(ix/\hbar-1/p_0)p}\ dp\right]$$

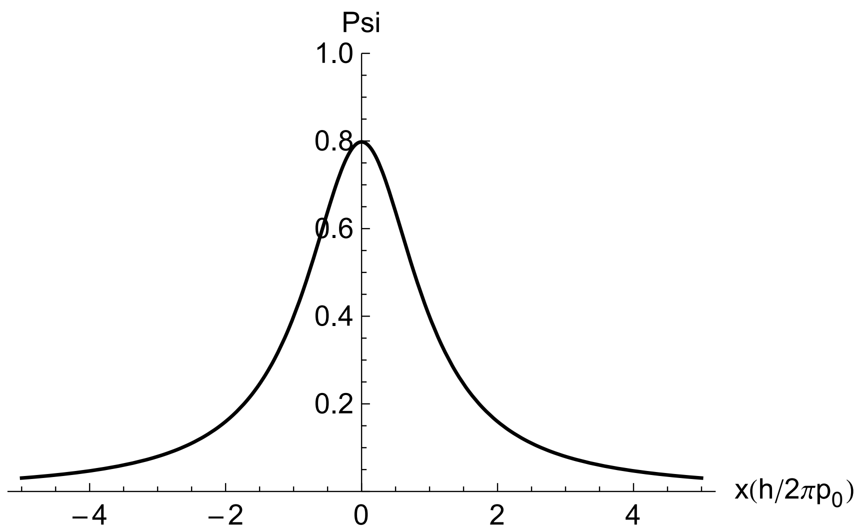
$$=\frac{1}{\sqrt{2\pi\hbar p_0}}\left\{\left[\frac{e^{(ix/\hbar+1/p_0)p}}{(ix/\hbar+1/p_0)}\right]_{-\infty}^{0}+\left[\frac{e^{(ix/\hbar-1/p_0)p}}{(ix/\hbar-1/p_0)}\right]_{0}^{\infty}\right\}$$

$$=\frac{1}{\sqrt{2\pi\hbar p_0}}\left\{\frac{1}{(ix/\hbar+1/p_0)}-\frac{1}{(ix/\hbar-1/p_0)}\right\}$$

$$=\frac{1}{\sqrt{2\pi\hbar p_0}}\left\{-\frac{(ix/\hbar-1/p_0)}{(x^2/\hbar^2+1/p_0^2)}+\frac{(ix/\hbar+1/p_0)}{(x^2/\hbar^2+1/p_0^2)}\right\}$$

$$=\sqrt{\frac{2\hbar^3}{\pi p_0^3}}\frac{1}{x^2+\frac{\hbar^2}{p_0^2}}$$

Shown below



c) Find the wave function at time *t* by adding in the Schrödinger time-evolution phase to each energy eigenstate. Momentum eigenstates are also energy eigenstates, so the wave function is already in energy state superposition format:

$$\psi(x)=\langle x|\psi\rangle=\int_{-\infty}^{\infty}\langle x|p\rangle\langle p|\psi\rangle dp=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{ipx/\hbar}\psi(p)dp$$

From the results above we have $\psi(p)=e^{-|p|/p_0}\big/\sqrt{p_0}$, so the time dependent wave function is

$$\psi(x,t)=\frac{1}{\sqrt{2\pi\hbar p_0}}\int_{-\infty}^{\infty}e^{-|p|/p_0}\ e^{ipx/\hbar}\ e^{-ip^2t/2m\hbar}\ dp$$

Now use this to find the momentum space wave function as before

$$\psi(p,t)=\langle p|\psi\rangle=\int_{-\infty}^{\infty}\langle p|x\rangle\langle x|\psi(t)\rangle dx=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{-ipx/\hbar}\psi(x,t)dx$$
$$=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{-ipx/\hbar}A\int_{-\infty}^{\infty}e^{-|p'|/p_0}\;e^{ip'x/\hbar}e^{-ip'^2t/2m\hbar}\;dp'\,dx$$
$$=\frac{A}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{-|p'|/p_0}e^{-ip'^2t/2m\hbar}\left\{\int_{-\infty}^{\infty}\;e^{i(p'-p)x/\hbar}\;dx\right\}dp'$$
$$=\frac{1}{2\pi\hbar\sqrt{p_0}}\int_{-\infty}^{\infty}e^{-|p'|/p_0}e^{-ip'^2t/2m\hbar}\left\{2\pi\hbar\delta(p'-p)\right\}dp'$$
$$=\frac{1}{\sqrt{p_0}}e^{-|p|/p_0}e^{-ip^2t/2m\hbar}$$

meaning simply that each momentum component acquires the relevant Schrödinger time-evolution phase. Because the probability density in momentum space is just the complex square of this, the phase disappears, giving

$$\mathcal{P}(p=-p_1\to p_1,t)=\int_{-p_1}^{p_1}|\psi(p,t)|^2\,dp=\int_{-p_1}^{p_1}|\psi(p,0)|^2\,dp$$

The momentum measurement is time independent, which is what we expect for a free particle because there are no external forces.

5. Bra-ket practice

a) $$\psi(x)=\langle x|\psi\rangle$$

b) $$\psi(p)=\langle p|\psi\rangle$$

c) $$|\psi\rangle\doteq\begin{pmatrix}\langle 1|\psi\rangle\\ \langle 2|\psi\rangle\\ \langle 3|\psi\rangle\\ \vdots\end{pmatrix}=\begin{pmatrix}c_1\\ c_2\\ c_3\\ \vdots\end{pmatrix}$$

d) $$\psi(x)=\langle x|\psi\rangle=\int_{-\infty}^{\infty}\langle x|p\rangle\langle p|\psi\rangle dp=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{ipx/\hbar}\psi(p)dp$$

e) $$\mathcal{P}(x)=|\langle x|\psi\rangle|^2=|\psi(x)|^2$$

f) $$\mathcal{P}(p)=|\langle p|\psi\rangle|^2=|\psi(p)|^2$$

g) $$\mathcal{P}(a\le x\le b)=\int_a^b|\psi(x)|^2\,dx$$

h) $$\mathcal{P}(p_1\le p\le p_2)=\int_{p_1}^{p_2}|\psi(p)|^2\,dp$$

i) $$1=\langle\psi|\psi\rangle=\int_{-\infty}^{\infty}\langle\psi|x\rangle\langle x|\psi\rangle dx=\int_{-\infty}^{\infty}|\psi(x)|^2\,dx=\begin{pmatrix}c_1^* & c_2^* & c_3^* & \cdots\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\\ \vdots\end{pmatrix}=\sum_i|c_i|^2$$

j)

$$\langle\phi|\psi\rangle = \int_{-\infty}^{\infty}\langle\phi|x\rangle\langle x|\psi\rangle\,dx = \int_{-\infty}^{\infty}\phi^*(x)\psi(x)\,dx = \begin{pmatrix} b_1^* & b_2^* & b_3^* & \cdots \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} = \sum_i b_i^* c_i$$

k)

$$\mathcal{P}(E_n) = \left|\langle E_n|\psi\rangle\right|^2 = \left|\int_{-\infty}^{\infty}\langle E_n|x\rangle\langle x|\psi\rangle\,dx\right|^2 = \left|\int_{-\infty}^{\infty}\psi_{E_n}^*(x)\psi(x)\,dx\right|^2$$

$$= \left|\begin{pmatrix} 0 & 0 & 1 & \cdots \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}\right|^2 = |c_n|^2$$