1.6.1 The matrix for the operator Ω is

$$\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

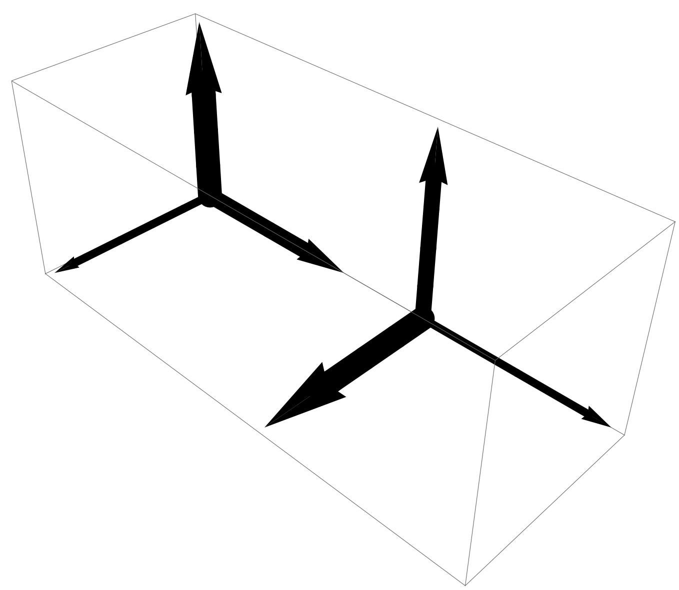
Its action on the unit vectors $|i\rangle$ is

$$\Omega|1\rangle \doteq \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \doteq |2\rangle$$

$$\Omega|2\rangle \doteq \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \doteq |3\rangle$$

$$\Omega|3\rangle \doteq \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \doteq |1\rangle$$

If we interpret these unit vectors as representing the spatial unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, then Ω transforms the $\hat{\mathbf{i}}\,\hat{\mathbf{j}}\,\hat{\mathbf{k}}$ triad into the $\hat{\mathbf{j}}\,\hat{\mathbf{k}}\,\hat{\mathbf{i}}$ triad. The diagram shows the two coordinate systems with one displaced for clarity. To get from the $\hat{\mathbf{i}}\,\hat{\mathbf{j}}\,\hat{\mathbf{k}}$ triad to the $\hat{\mathbf{j}}\,\hat{\mathbf{k}}\,\hat{\mathbf{i}}$ triad requires a rotation of $2\pi/3 = 120^\circ$ around the $\langle 111\rangle$ direction (i.e., the diagonal of the unit cube). To confirm this, find the eigenvectors of the matrix. The $\langle 111\rangle$ vector is the only real eigenvector.



1.8.2 (a) The matrix is

$$\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Its adjoint is (complex transpose)

$$\Omega^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$\Omega = \Omega^\dagger$, so the matrix is Hermitian.

(b) To find the eigenvalues and eigenvectors, first find the characteristic equation:

$$\begin{pmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad -\lambda\left(\lambda^2 - 0\right) + 1\left(0 + \lambda\right) = 0$$

$$\Rightarrow \quad -\lambda\left(\lambda^2 - 1\right) = 0$$

$$\Rightarrow \quad \lambda = 0,1,-1$$

Now find each eigenvector

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 1\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} w = u \\ 0 = v \\ u = w \end{matrix} \quad \Rightarrow \quad v = 0, u = w$$

$$|u|^2 + |v|^2 + |w|^2 = 1 \quad \Rightarrow \quad 2|u|^2 = 1 \quad \Rightarrow \quad u = \tfrac{1}{\sqrt{2}}, v = 0, w = \tfrac{1}{\sqrt{2}} \quad \Rightarrow \quad |1\rangle \doteq \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} w = 0 \\ 0 = 0 \\ u = 0 \end{matrix} \quad \Rightarrow \quad u = w = 0$$

$$|u|^2 + |v|^2 + |w|^2 = 1 \quad \Rightarrow \quad |v|^2 = 1 \quad \Rightarrow \quad u = 0, v = 1, w = 0 \quad \Rightarrow \quad |0\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -1\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} w = -u \\ 0 = v \\ u = -w \end{matrix} \quad \Rightarrow \quad v = 0, u = -w$$

$$|u|^2 + |v|^2 + |w|^2 = 1 \quad \Rightarrow \quad 2|u|^2 = 1 \quad \Rightarrow \quad u = \tfrac{1}{\sqrt{2}}, v = 0, w = \tfrac{-1}{\sqrt{2}} \quad \Rightarrow \quad |-1\rangle \doteq \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

(c) The transformation matrix is built from the eigenvectors as columns:

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Now transform the original matrix:

$$\begin{aligned} U^\dagger \Omega U &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Yielding a diagonal matrix with the eigenvalues along the diagonal.

1.8.10 The commutator is

$$[\Omega,\Lambda] = \Omega\Lambda - \Lambda\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

so they do commute.
Find the eigenvalues and eigenvectors of Ω:

$$\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = 0 \quad \Rightarrow -\lambda(1-\lambda)^2 + 1(0+\lambda) = 0$$

$$\Rightarrow -\lambda\left(\lambda^2 - 2\lambda\right) = 0 \quad \Rightarrow -\lambda^2(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 0,0,2$$

Now find each eigenvector

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 2\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} u+w=2u \\ 0=2v \\ u+w=2w \end{matrix} \quad \Rightarrow \quad v=0, u=w$$

$$|u|^2 + |v|^2 + |w|^2 = 1 \quad \Rightarrow \quad 2|u|^2 = 1 \quad \Rightarrow \quad u = \tfrac{1}{\sqrt{2}}, v=0, w=\tfrac{1}{\sqrt{2}} \quad \Rightarrow \quad |2\rangle \doteq \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} u+w=0 \\ 0=0 \\ u+w=0 \end{matrix} \quad \Rightarrow \quad u=-w, v=?$$

$$|u|^2 + |v|^2 + |w|^2 = 1 \quad \Rightarrow \quad 2|u|^2 = 1 \quad \Rightarrow \quad u = \tfrac{1}{\sqrt{2}}, v=0, w=\tfrac{-1}{\sqrt{2}} \quad \Rightarrow \quad |0\rangle_1 \doteq \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

The 0 eigenvalue is degenerate, which implies ambiguity in the choice of eignevectors. So let's diagonalize the Λ matrix and see if that decides which basis we should choose.

$$\begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix} = 0$$

$$\Rightarrow\ (2-\lambda)\left[-\lambda(2-\lambda)-1\right]-1\left[(2-\lambda)+1\right]+1\left[-1+\lambda\right]=0$$

$$\Rightarrow\ \lambda^3-4\lambda^2+\lambda+6=0 \quad \Rightarrow\ (\lambda-3)(\lambda-2)(\lambda+1)=0$$

$$\Rightarrow\ \lambda = 3,2,-1$$

(The factors of 6 are 3,2,1, so we expect a simple solution to involve those values (pos or neg)). Now find the non-degenerate eigenvectors:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 3\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad\Rightarrow\quad \begin{matrix} 2u+v+w=3u \\ u-w=3v \\ u-v+2w=3w \end{matrix} \quad\Rightarrow\quad v=0, u=w$$

$$|u|^2+|v|^2+|w|^2=1 \ \Rightarrow\ 2|u|^2=1 \ \Rightarrow\ u=\tfrac{1}{\sqrt{2}}, v=0, w=\tfrac{1}{\sqrt{2}} \ \Rightarrow\ |3\rangle \doteq \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 2\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad\Rightarrow\quad \begin{matrix} 2u+v+w=2u \\ u-w=2v \\ u-v+2w=2w \end{matrix} \quad\Rightarrow\quad u=v=-w$$

$$|u|^2+|v|^2+|w|^2=1 \ \Rightarrow\ 3|u|^2=1 \ \Rightarrow\ u=\tfrac{1}{\sqrt{3}}, v=\tfrac{1}{\sqrt{3}}, w=\tfrac{-1}{\sqrt{3}} \ \Rightarrow\ |2\rangle \doteq \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -1\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad\Rightarrow\quad \begin{matrix} 2u+v+w=-u \\ u-w=-v \\ u-v+2w=-w \end{matrix} \quad\Rightarrow\quad u=-\frac{v}{2}=-w$$

$$|u|^2+|v|^2+|w|^2=1 \ \Rightarrow\ 6|u|^2=1 \ \Rightarrow\ u=\tfrac{1}{\sqrt{6}}, v=\tfrac{-2}{\sqrt{6}}, w=\tfrac{-1}{\sqrt{6}} \ \Rightarrow\ |-1\rangle \doteq \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}$$

The transformation matrix is built from the eigenvectors as columns:

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

Now transform the original matrices:

$$U^{\dagger}\Omega U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U^{\dagger}\Lambda U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{2}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{3}{\sqrt{2}} & \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Yielding diagonal matrices with the eigenvalues along the diagonals.