This is the relationship that must hold among the parameters for a ray going from S to P by way of refraction at the spherical interface. Although this expression is exact, it is rather complicated. If A is moved to a new location by changing φ, the new ray will not intercept the optical axis at P. (See Problem 5.1 concerning the Cartesian oval, which is the interface configuration that would bring any ray, regardless of φ, to P.) The approximations that are used to represent ℓ_o and ℓ_i, and thereby simplify Eq. (5.5), are crucial in all that is to follow. Recall that

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \cdots \tag{5.6}$$

and

$$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \cdots \tag{5.7}$$

If we assume small values of φ (i.e., A close to V), $\cos\varphi \approx 1$. Consequently, the expressions for ℓ_o and ℓ_i yield $\ell_o \approx s_o$, $\ell_i \approx s_i$, and to that approximation

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} \tag{5.8}$$

We could have begun this derivation with Snell's Law rather than Fermat's Principle (Problem 5.5), in which case small values of φ would have led to $\sin\varphi \approx \varphi$ and Eq. (5.8) once again. This approximation delineates the domain of what is called *first-order theory*; we'll examine *third-order theory* ($\sin\varphi \approx \varphi - \varphi^3/3!$) in the next chapter. Rays that arrive at shallow angles with respect to the optical axis (such that φ and h are appropriately small) are known as **paraxial rays**. The *emerging wavefront segment corresponding to these paraxial rays is essentially spherical and will form a "perfect" image at its center P located at* s_i. Notice that Eq. (5.8) is independent of the location of A over a small area about the symmetry axis, namely, the *paraxial region*. Gauss, in 1841, was the first to give a systematic exposition of the formation of images under the above approximation, and the result is variously known as *first-order*, *paraxial*, or **Gaussian Optics**. It soon became the basic theoretical tool by which lenses would be designed for several decades to come. If the optical system is well corrected, an incident spherical wave will emerge in a form very closely resembling a spherical wave. Consequently, as the perfection of the system increases, it more closely approaches first-order theory. Deviations from that of paraxial analysis will provide a convenient measure of the quality of an actual optical device.

If point-F_o in Fig. 5.8 is imaged at infinity ($s_i = \infty$), we have

$$\frac{n_1}{s_o} + \frac{n_2}{\infty} = \frac{n_2 - n_1}{R}$$

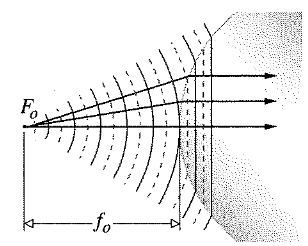


Figure 5.8 Plane waves propagating beyond a spherical interface—the object focus.

That special object distance is defined as the **first focal length** or the **object focal length**, $s_o \equiv f_o$, so that

$$f_o = \frac{n_1}{n_2 - n_1} R \tag{5.9}$$

Point-F_o is known as the **first** or **object focus**. Similarly, the **second** or **image focus** is the axial point-F_i, where the image is formed when $s_o = \infty$; that is,

$$\frac{n_1}{\infty} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

Defining the **second** or **image focal length** f_i as equal to s_i in this special case (Fig. 5.9), we have

$$f_i = \frac{n_2}{n_2 - n_1} R \tag{5.10}$$

Recall that an image is virtual when the rays diverge from it (Fig. 5.10). Analogously, ***an object is virtual when the rays converge toward it*** (Fig. 5.11). Observe that the virtual object is now on the right-hand side of the vertex, and therefore s_o will be a negative quantity. Moreover, the surface is concave, and its radius will also be negative, as required by Eq. (5.9), since f_o would be negative. In the same way, the virtual image distance appearing to the left of V is negative.

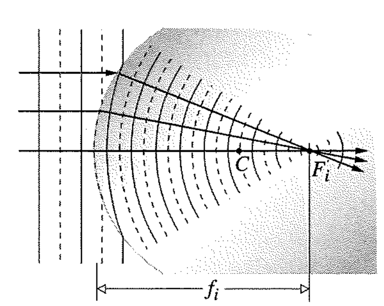


Figure 5.9 The reshaping of plane into spherical waves at a spherical interface—the image focus.

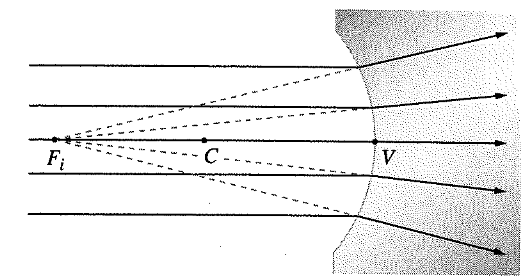


Figure 5.10 A virtual image point.

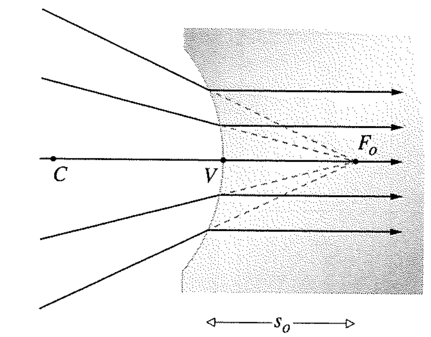


Figure 5.11 A virtual object point.

EXAMPLE 5.2

A long horizontal flint-glass ($n_g = 1.800$) cylinder is 20.0 cm in diameter and has a convex hemispherical left end ground and polished onto it. The device is immersed in ethyl alcohol ($n_a = 1.361$) and a tiny LED is located on the central axis in the liquid 80.0 cm to the left of the vertex of the hemisphere. Locate the image of the LED. What would happen if the alcohol was replaced by air?

SOLUTION

Return to Eq. (5.8),

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

Here $n_1 = 1.361$, $n_2 = 1.800$, $s_o = +80.0$ cm, and $R = +10.0$ cm. We can work the problem in centimeters, whereupon the equation becomes

$$\frac{1.361}{80.0} + \frac{1.800}{s_i} = \frac{1.800 - 1.361}{10.0}$$

$$\frac{1.800}{s_i} = \frac{0.439}{10} - \frac{1.361}{80}$$

$$1.800 = (0.0439 - 0.01701)s_i$$

$$s_i = 66.9 \text{ cm}$$

With the alcohol in place the image is within the glass, 66.9 cm to the right of the vertex ($s_i > 0$). Removing the liquid,

$$\frac{1}{80.0} + \frac{1.800}{s_i} = \frac{0.800}{10.0}$$

and

$$s_i = 26.7 \text{ cm}$$

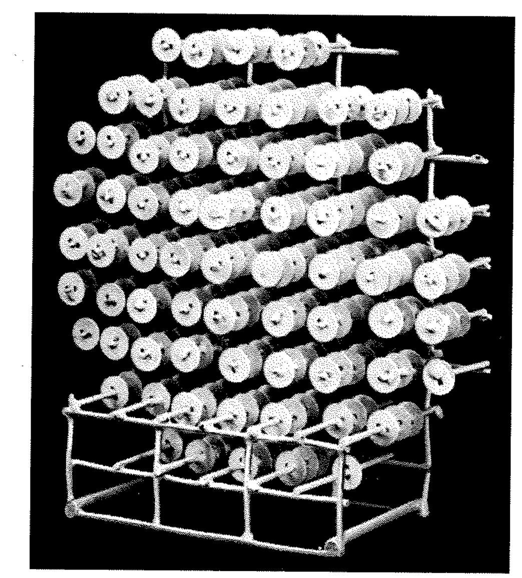
The refraction at the interface depends on the ratio (n_2/n_1) of the two indices. The bigger is $(n_2 - n_1)$, the smaller will be s_i.

5.2.3 Thin Lenses

Lenses are made in a wide range of forms; for example, there are acoustic and microwave lenses. Some of the latter are made of glass or wax in easily recognizable shapes, whereas others are far more subtle in appearance (see photo). Most often a lens has two or more refracting interfaces, and at least one of these is curved. Generally, the nonplanar surfaces are centered on a common axis. These surfaces are most frequently spherical segments and are often coated with thin dielectric films to control their transmission properties (see Section 9.9).

A lens that consists of one element (i.e., it has only two refracting surfaces) is a *simple lens*. The presence of more than one element makes it a *compound lens*. A lens is also classified as to whether it is *thin* or *thick*—that is, whether or not its thickness is effectively negligible. We will limit ourselves, for the most part, to *centered systems* (for which all surfaces are rotationally symmetric about a common axis) of spherical surfaces. Under these restrictions, the simple lens can take the forms shown in Fig. 5.12.

Lenses that are variously known as ***convex***, ***converging***, or ***positive*** are thicker at the center and so tend to decrease the radius of curvature of the wavefronts. In other words, the incident wave converges more as it traverses the lens, assuming, of course, that the index of the lens is greater than that of the



A lens for short-wavelength radiowaves. The disks serve to refract these waves much as rows of atoms refract light. (Optical Society of America)

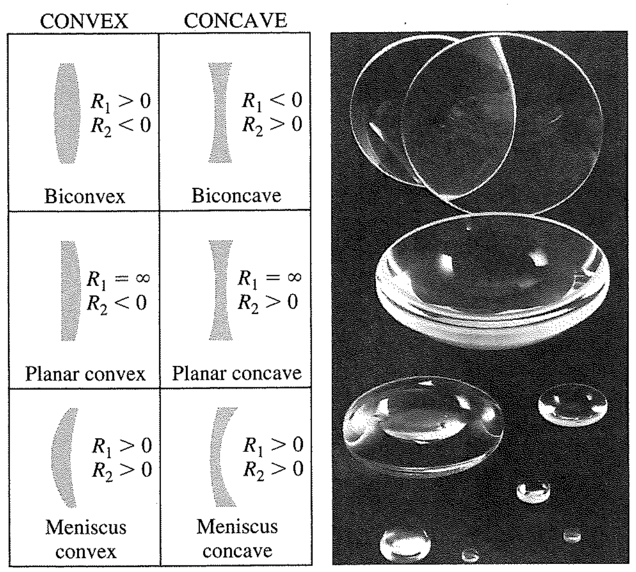
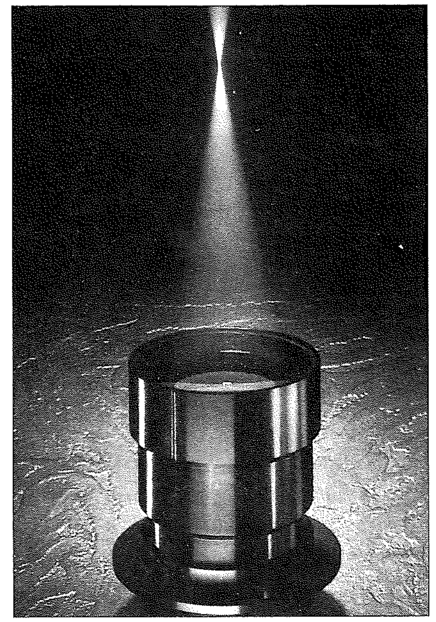


Figure 5.12 Cross sections of various centered spherical simple lenses. The surface on the left is $\neq 1$, since it is encountered first. Its radius is R_1. (Melles Griot)



A lens focusing a beam of light. (L-3 Communications Tinsley Labs Inc.)

media in which it is immersed. ***Concave***, ***diverging***, or ***negative*** lenses, on the other hand, are thinner at the center and tend to advance that portion of the incident wavefront, causing it to diverge more than it did prior to entry.

Thin-Lens Equations

Return to the discussion of refraction at a single spherical interface, where the location of the conjugate points-S and -P is given by

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} \qquad [5.8]$$

When s_o is large for a fixed $(n_2 - n_1)/R$, s_i is relatively small. The cone of rays from S has a small central angle, the rays do not diverge very much, and the refraction at the interface can cause them all to converge at P. As s_o decreases, the ray-cone angle increases, the divergence of the rays increases, and s_i moves away from the vertex; that is, both θ_i and θ_t increase until finally $s_o = f_o$ and $s_i = \infty$. At that point, $n_1/s_o = (n_2 - n_1)/R$, so that if s_o gets any smaller, s_i will have to be negative, if Eq. (5.8) is to hold. In other words, the image becomes virtual (Fig. 5.13).

Let's now locate the conjugate points for a lens of index n_l surrounded by a medium of index n_m, as in Fig. 5.14, where another end has simply been ground onto the piece in Fig. 5.13*c*. This certainly isn't the most general set of circumstances, but it is the most common, and even more cogently, it is the simplest.* We know from Eq. (5.8) that the paraxial rays issuing from S at s_{o1} will appear to meet at P', a distance, which we now call s_{i1}, from V_1, given by

$$\frac{n_m}{s_{o1}} + \frac{n_l}{s_{i1}} = \frac{n_l - n_m}{R_1} \qquad (5.11)$$

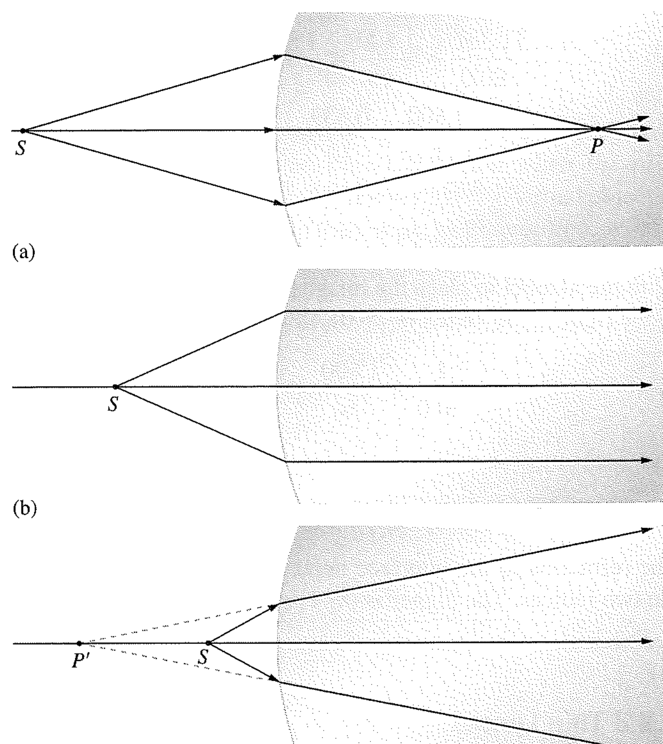


Figure 5.13 Refraction at a spherical interface between two transparent media shown in cross section.

*See Jenkins and White, *Fundamentals of Optics*, p. 57, for a derivation containing three different indices.

Thus, as far as the second surface is concerned, it "sees" rays coming toward it from P', which serves as its object point a distance s_{o2} away. Furthermore, the rays arriving at that second surface are in the medium of index n_l. The object space for the second interface that contains P' therefore has an index n_l. Note that the rays from P' to that surface are indeed straight lines. Considering the fact that

$$|s_{o2}| = |s_{i1}| + d$$

since s_{o2} is on the left and therefore positive, $s_{o2} = |s_{o2}|$, and s_{i1} is also on the left and therefore negative, $-s_{i1} = |s_{i1}|$, we have

$$s_{o2} = -s_{i1} + d \qquad (5.12)$$

At the second surface Eq. (5.8) yields

$$\frac{n_l}{(-s_{i1} + d)} + \frac{n_m}{s_{i2}} = \frac{n_m - n_l}{R_2} \qquad (5.13)$$

Here $n_l > n_m$ and $R_2 < 0$, so that the right-hand side is positive. Adding Eqs. (5.11) and (5.13), we have

$$\frac{n_m}{s_{o1}} + \frac{n_m}{s_{i2}} = (n_l - n_m)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{n_l d}{(s_{i1} - d)s_{i1}} \qquad (5.14)$$

If the lens is thin enough ($d \to 0$), the last term on the right is effectively zero. As a further simplification, assume the surrounding medium to be air (i.e., $n_m \approx 1$). Accordingly, we have the very useful **Thin-Lens Equation**, often referred to as the **Lensmaker's Formula**:

$$\frac{1}{s_o} + \frac{1}{s_i} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (5.15)$$

where we let $s_{o1} = s_o$ and $s_{i2} = s_i$. The points-V_1 and -V_2 tend to coalesce as $d \to 0$, so that s_o and s_i can be measured from either the vertices or the lens center.

Just as in the case of the single spherical surface, if s_o is moved out to infinity, the image distance becomes the focal length f_i, or symbolically,

$$\lim_{s_o \to \infty} s_i = f_i$$

Similarly

$$\lim_{s_i \to \infty} s_o = f_o$$

It is evident from Eq. (5.15) that for a thin lens $f_i = f_o$, and consequently we drop the subscripts altogether. Thus

$$\frac{1}{f} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (5.16)$$

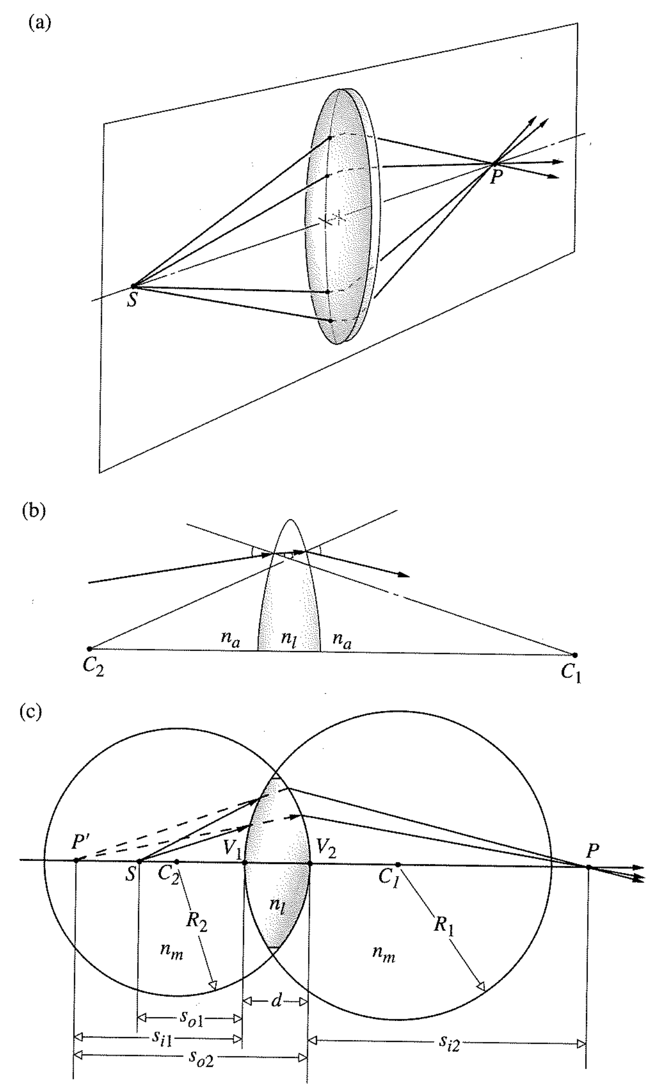


Figure 5.14 A spherical lens. (a) Rays in a vertical plane passing through a lens. Conjugate foci. (b) Refraction at the interfaces where the lens is immersed in air and $n_m = n_a$. The radius drawn from C_1 is normal to the first surface, and as the ray enters the lens it bends down *toward* that normal. The radius from C_2 is normal to the second surface; and as the ray emerges, since $n_l > n_a$, the ray bends down *away* from that normal. (c) The geometry.

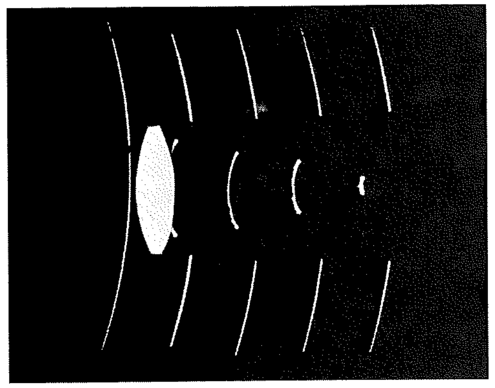
and

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} \qquad (5.17)$$

which is the famous **Gaussian Lens Formula** (see photo).

As an example of how these expressions might be used, let's compute the focal length in air of a thin planar-convex lens having a radius of curvature of 50 mm and an index of 1.5. With light entering on the planar surface ($R_1 = \infty$, $R_2 = -50$),

$$\frac{1}{f} = (1.5 - 1)\left(\frac{1}{\infty} - \frac{1}{-50}\right)$$



The actual wavefronts of a diverging lightwave partially focused by a lens. The photo shows five exposures, each separated by about 100 ps (i.e., 100×10^{-12} s), of a spherical pulse 10 ps long as it swept by and through a converging lens. The picture was made using a holographic technique. (N.H. Abramson)

whereas if instead it arrives at the curved surface ($R_1 = +50$, $R_2 = \infty$),

$$\frac{1}{f} = (1.5 - 1)\left(\frac{1}{+50} - \frac{1}{\infty}\right)$$

and in either case $f = 100$ mm. If an object is alternately placed at distances 600 mm, 200 mm, 150 mm, 100 mm, and 50 mm from the lens on either side, we can find the image points from Eq. (5.17). First, with $s_o = 600$ mm

$$s_i = \frac{s_o f}{s_o - f} = \frac{(600)(100)}{600 - 100}$$

and $s_i = 120$ mm. Similarly, the other image distances are 200 mm, 300 mm, ∞, and -100 mm, respectively.

Interestingly enough, when $s_o = \infty$, $s_i = f$; as s_o decreases, s_i increases positively until $s_o = f$ and s_i is negative thereafter. Figure 5.15 shows this behavior pictorially. The lens is capable of adding a certain amount of convergence to the rays. As the divergence of the incident light increases, the lens is less able to pull the rays together and point-P moves farther to the right.

You can qualitatively check this out with a simple convex lens and a small electric light—the high-intensity variety is probably the most convenient. Standing as far as you can from the source, project a clear image of it onto a white sheet of paper. You should be able to see the lamp quite clearly and not just as a blur. That image distance approximates f. Now move the lens in toward S, adjusting s_i to produce a clear image. It will surely increase. As $s_o \rightarrow f$, a clear image of the lamp can be projected, but only on an increasingly distant screen. For $s_o < f$, there will just be a blur where the farthest wall intersects the diverging cone of rays—the image is virtual.

Focal Points and Planes

Figure 5.16 summarizes some of the situations described analytically by Eq. 5.16. Observe that if a lens of index n_l is immersed in a medium of index n_m,

$$\frac{1}{f} = (n_{lm} - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{5.18}$$

The focal lengths in (a) and (b) of Fig. 5.16 are equal, because the same medium exists on either side of the lens. Since $n_l > n_m$, it follows that $n_{lm} > 1$. In both cases $R_1 > 0$ and $R_2 < 0$, so that each focal length is positive. We have a real object in (a) and a real image in (b). In (c), $n_l < n_m$, and consequently f is negative. In (d) and (e), $n_{lm} > 1$ but $R_1 < 0$, whereas $R_2 > 0$, so f is again negative, and the object in one case and the image in the other are virtual. In (f), $n_{lm} < 1$, yielding an $f > 0$.

Notice that in each instance it is particularly convenient to draw a ray through the center of the lens, which, because it is perpendicular to both surfaces, is undeviated. Suppose, instead, that an off-axis paraxial ray emerges from the lens parallel to its incident direction, as in Fig. 5.17. We maintain that all such rays

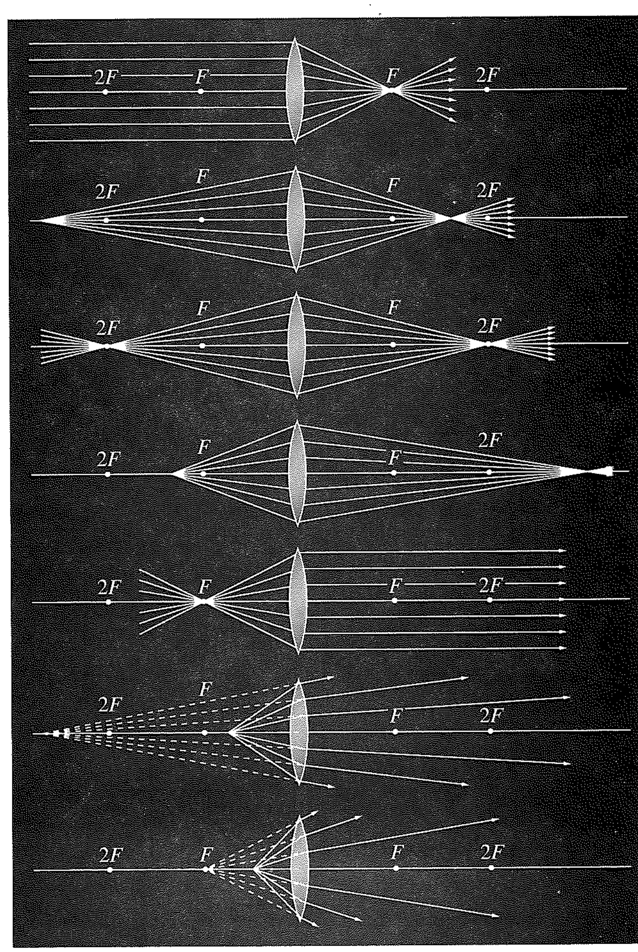


Figure 5.15 Conjugate object and image points for a thin convex lens.

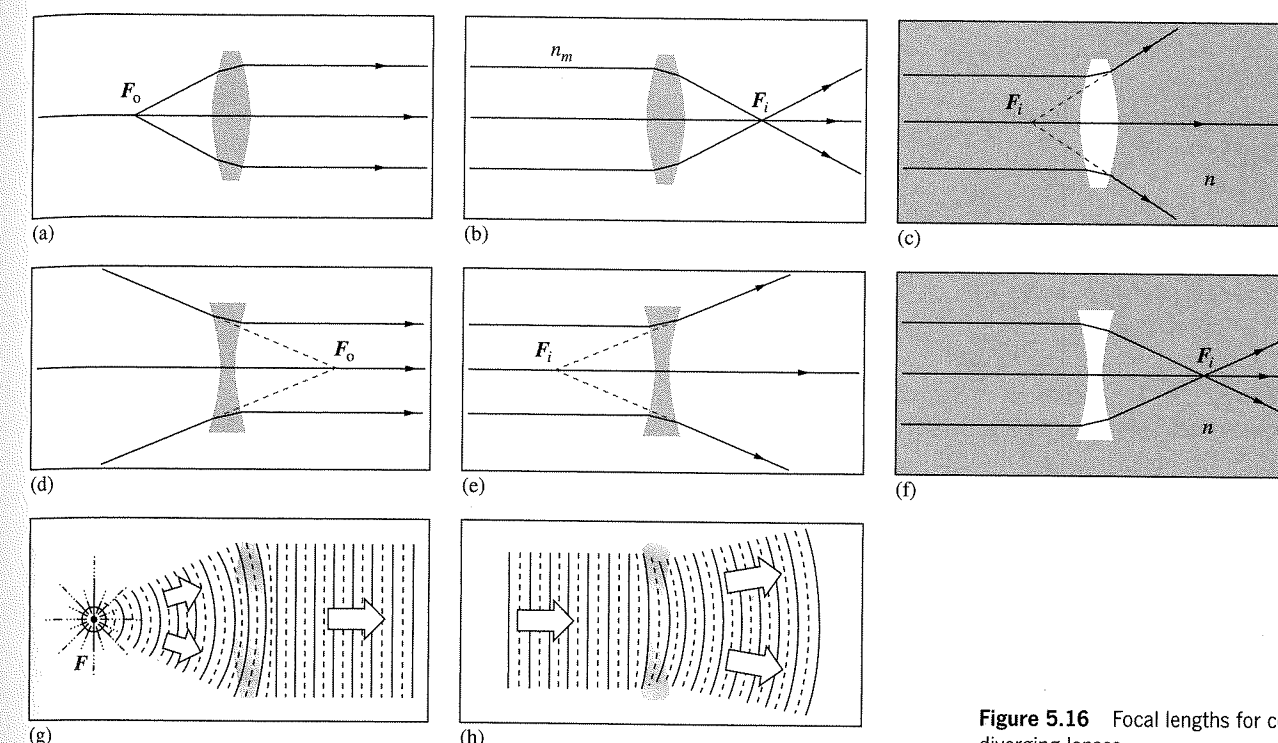


Figure 5.16 Focal lengths for converging and diverging lenses.

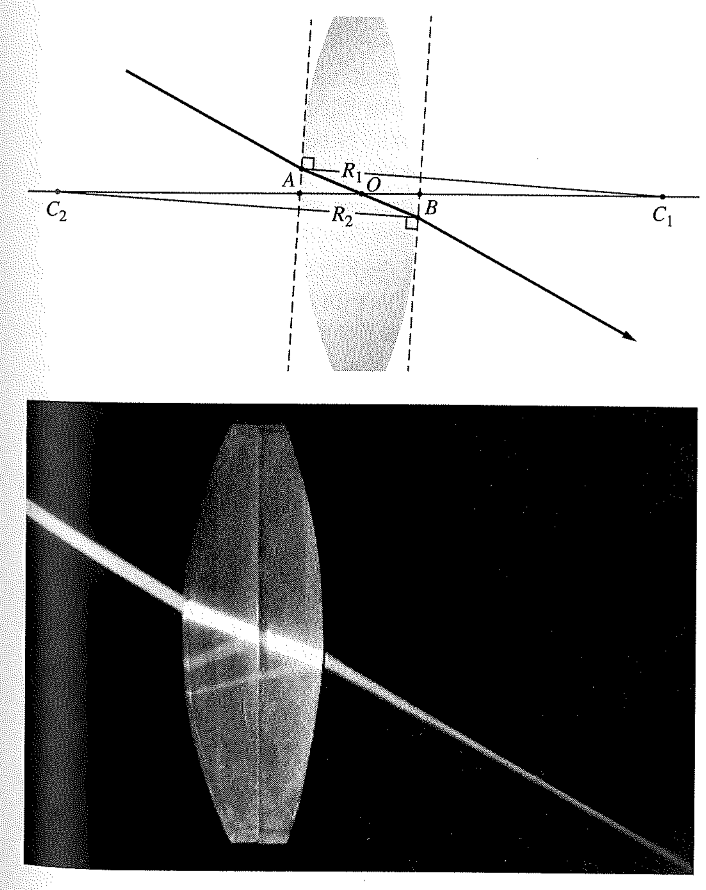


Figure 5.17 The optical center of a lens. (E.H.)

will pass through the point defined as the **optical center** O of the lens. To see this, draw two parallel planes, one on each side tangent to the lens at any pair of points-A and -B. This can easily be done by selecting A and B such that the radii $\overline{AC_1}$ and $\overline{BC_2}$ are themselves parallel. It is to be shown that the paraxial ray traversing $\overline{AB}$ enters and leaves the lens in the same direction. It's evident from the diagram that triangles AOC_1 and BOC_2 are similar, in the geometric sense, and therefore their sides are proportional. Hence, $|R_1|(\overline{OC_2}) = |R_2|(\overline{OC_1})$, and since the radii are constant, the location of O is constant, independent of A and B. As we saw earlier (Problem 4.38 and Fig. P.4.38), a ray traversing a medium bounded by parallel planes will be displaced laterally but will suffer no angular deviation. This displacement is proportional to the thickness, which for a thin lens is negligible. ***Rays passing through O may, accordingly, be drawn as straight lines***. It is customary when dealing with thin lenses simply to place O midway between the vertices.

Recall that a bundle of parallel paraxial rays incident on a spherical refracting surface comes to a focus at a point on the optical axis (Fig. 5.10). As shown in Fig. 5.18, this implies that several such bundles entering in a narrow cone will be focused on a spherical segment σ, also centered on C. The undeviated rays normal to the surface, and therefore passing through C, locate the foci on σ. Since the ray cone must indeed be narrow, σ can satisfactorily be represented as a plane normal to the symmetry axis and passing through the image focus. It is known as a **focal plane**. In the same way, limiting ourselves to paraxial theory, a lens will

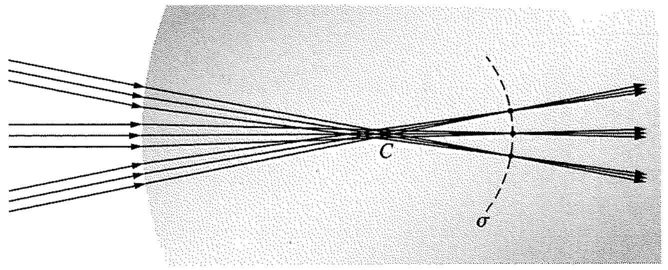


Figure 5.18 Focusing of several ray bundles.

focus all incident parallel bundles of rays* onto a surface called the **second** or **back focal plane**, as in Fig. 5.19. Here each point on σ is located by the undeviated ray through O. Similarly, the **first** or **front focal plane** contains the object focus F_o.

There is another practical observation about lenses that's worth introducing before we move on, and that concerns the relationship between shape and focal length. Return to Eq. (5.16), which deals with the physical characteristics of a lens and, for simplicity, consider an equiconvex lens for which $R_1 = -R_2 = R$. The equation then becomes $f = R/2(n_l - 1)$ and we see immediately that the smaller the radius of the lens, that is, the squatter it is, the shorter will be its focal length. ***A nearly flat lens will have a long focal length***, whereas a small sphere (hardly a "thin lens") will have a tiny focal length. Of course, the greater the curvature $(1/R)$ of each interface, the greater the bending of the rays, as shown in Fig. 5.20. Also keep in mind that f is inversely proportional to n_l, a fact we'll come back to later on when dealing with aberrations. If having a flatter lens is desirable, one need only increase its index of refraction while increasing R, thereby leaving the focal length unchanged.

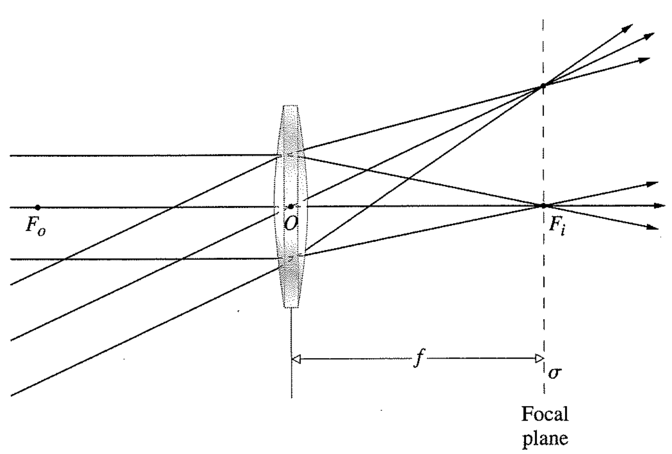


Figure 5.19 The focal plane of a lens.

*Perhaps the earliest literary reference to the focal properties of a lens appears in Aristophanes' play, *The Clouds*, which dates back to 423 B.C.E. In it Strepsiades plots to use a burning-glass to focus the Sun's rays onto a wax tablet and thereby melt out the record of a gambling debt.

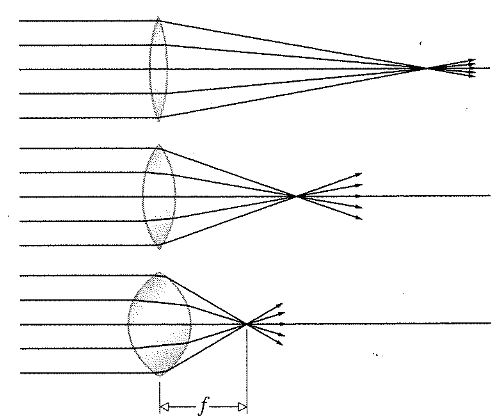
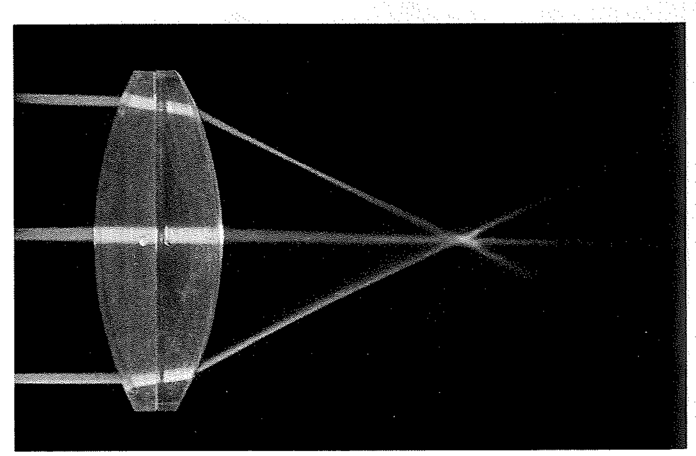


Figure 5.20 The greater the curvature $(1/R)$, the shorter the focal length.

Finite Imagery

Thus far we've treated the mathematical abstraction of a single-point source. Now let's deal with the fact that a great many such points combine to form a continuous finite object (Fig. 5.2). For the moment, imagine the object to be a segment of a sphere, σ_o, centered on C, as in Fig. 5.21. If σ_o is close to the spherical interface, point-S will have a virtual image P ($s_i < 0$ and therefore on the left of V). With S farther away, its image will be real ($s_i > 0$ and therefore on the right-hand side). In either case, each point on σ_o has a conjugate point on σ_i lying on a straight line through C. Within the restrictions of paraxial theory, these surfaces can be considered planar. Thus a small planar object normal to the optical axis will be imaged into a small planar region also normal to that axis. Note that if σ_o is moved out to infinity, the cone of rays from each source point will become **collimated** (i.e., parallel), and the image points will lie on the focal plane (Fig. 5.19).

By cutting and polishing the right side of the piece depicted in Fig. 5.21, we can construct a thin lens. Once again, the image (σ_i in Fig. 5.21) formed by the first surface of the lens will serve as the object for the second surface, which in turn will generate a final image. Suppose then that σ_i in Fig. 5.21*a* is the object for



Beams of light brought to a focus by a positive lens. (E.H.)

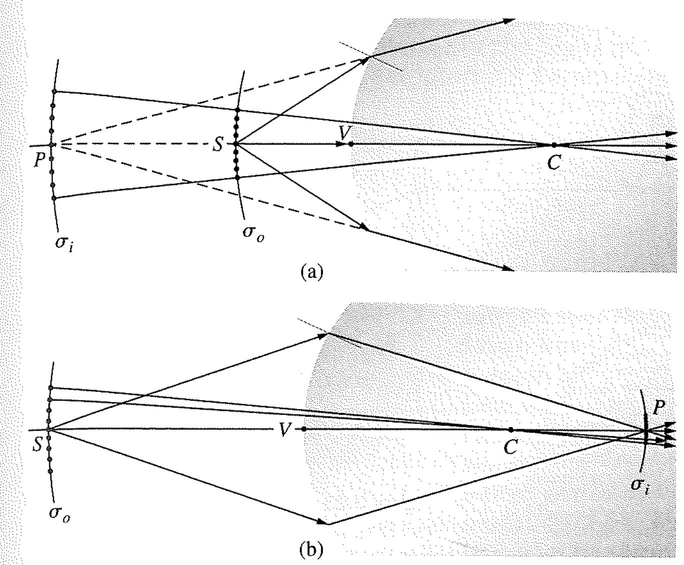


Figure 5.21 Finite imagery.

the second surface, which is assumed to have a negative radius. We already know what will happen—the situation is identical to that in Fig. 5.21*b* with the ray directions reversed. *The final image formed by a lens of a small planar object normal to the optical axis will itself be a small plane normal to that axis.*

The location, size, and orientation of an image produced by a lens can be determined, particularly simply, with ray diagrams. To find the image of the object in Fig. 5.22, we must locate the image point corresponding to each object point. Since all rays issuing from a source point in a paraxial cone will arrive at the image point, any two such rays will suffice to fix that point. Because we know the positions of the focal points, there are three rays that are especially easy to apply. The first (ray-1) is the undeviated ray through the center of the lens O. The other two (ray-2 and ray-3) make use of the fact that a ray passing through the focal point will emerge from the lens parallel to the central axis and vice versa. As a rule-of-thumb when sketching ray diagrams, draw the lens diameter (the vertical extent) roughly the size of the focal length. Then put in points on the central optical axis at one and two focal lengths, both in front of and behind the lens. You can usually locate the image by just tracing ray-1 and ray-2 from either the upper or lowermost points on the object.

Figure 5.23 shows how any *two* of these three rays locate the image of a point on the object. Incidentally, this technique dates back to the work of Robert Smith as long ago as 1738. This graphical procedure can be made even simpler by replacing the thin lens with a vertical plane perpendicular to the central axis passing through its center (Fig. 5.24). Presumably, if we were to extend every incoming ray forward a little and every outgoing ray backward a bit, each pair would meet on this plane. The total deviation of any ray can be envisaged as occurring all at once on that plane. This is equivalent to the actual process consisting of two separate angular shifts, one at each interface.

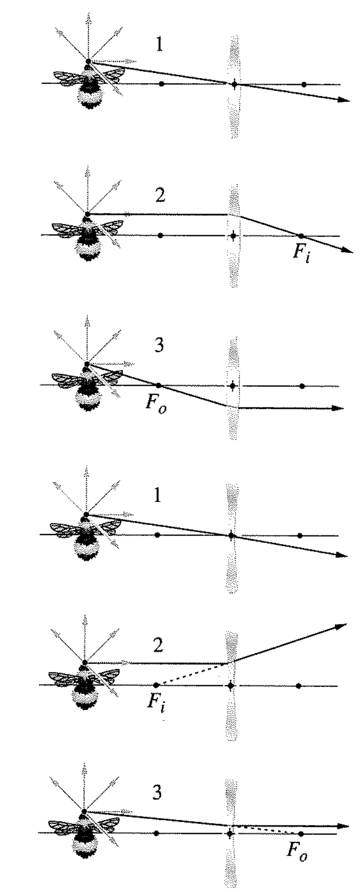


Figure 5.22 Tracing a few key rays through a positive and negative lens.

(As we'll see later, this is tantamount to saying that the two principal planes of a thin lens coincide.)

In accord with convention, transverse distances above the optical axis are taken as positive quantities, and those below the axis are given negative numerical values. Therefore in Fig. 5.24 $y_o > 0$ and $y_i < 0$. Here the image is said to be **inverted**, whereas if $y_i > 0$ when $y_o > 0$, it is **right-side-up** or **erect**. Observe that triangles AOF_i and $P_2P_1F_i$ are similar. Ergo

$$\frac{y_o}{|y_i|} = \frac{f}{(s_i - f)} \tag{5.19}$$

In the same way, triangles S_2S_1O and P_2P_1O are similar, and

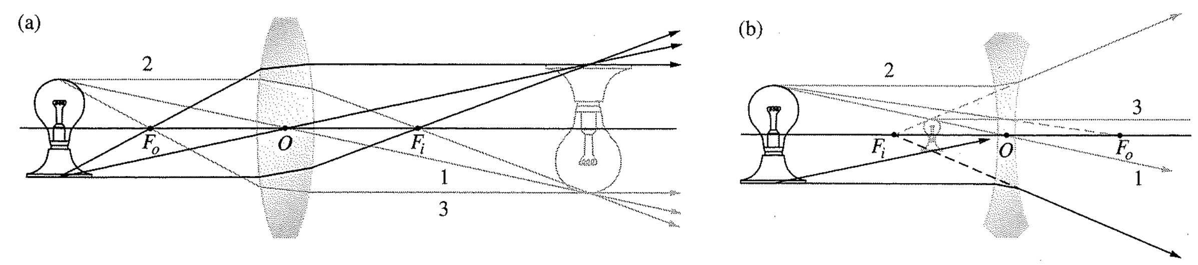
$$\frac{y_o}{|y_i|} = \frac{s_o}{s_i} \tag{5.20}$$

where all quantities other than y_i are positive. Hence

$$\frac{s_o}{s_i} = \frac{f}{(s_i - f)} \tag{5.21}$$

and

$$\frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_i}$$



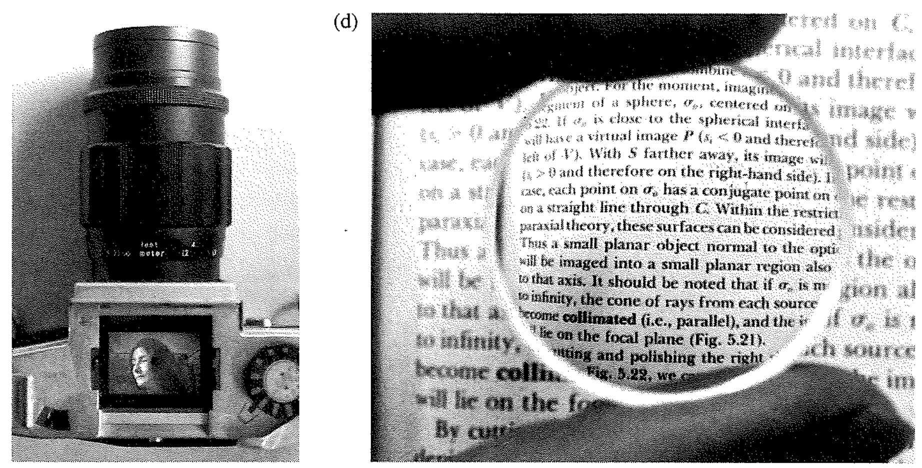


Figure 5.23 (a) A real object and a positive lens. (b) A real object and a negative lens. (c) A real image projected on the viewing screen of a 35-mm camera, much as the eye projects its image on the retina. Here a prism has been removed so you can see the image directly. (E.H.) (d) The minified, right-side-up, virtual image formed by a negative lens. (E.H.)

which is, of course, the Gaussian Lens Equation [Eq. (5.17)]. Furthermore, triangles $S_2S_1F_o$ and BOF_o are similar and

$$\frac{f}{(s_o - f)} = \frac{|y_i|}{y_o} \tag{5.22}$$

Using the distances measured from the focal points and combining this information with Eq. (5.19) leads to

$$x_o x_i = f^2 \tag{5.23}$$

This is the **Newtonian form** of the lens equation, the first statement of which appeared in Newton's *Opticks* in 1704. The signs of x_o and x_i are reckoned with respect to their concomitant foci. By convention, x_o is taken to be positive left of F_o, whereas x_i is positive on the right of F_i. It is evident from Eq. (5.23) that x_o and x_i have like signs, which means that ***the object and image must be on opposite sides of their respective focal points***. This is a good thing for the neophyte to remember when making those hasty freehand ray diagrams for which he or she is already infamous.

The ratio of the transverse dimensions of the final image formed by any optical system to the corresponding dimension of the object is defined as the *lateral* or **transverse magnification**, M_T, that is,

$$M_T \equiv \frac{y_i}{y_o} \tag{5.24}$$

Or from Eq. (5.20)

$$M_T = -\frac{s_i}{s_o} \tag{5.25}$$

A positive M_T connotes an erect image, while a negative value means the image is inverted (see Table 5.2). Bear in mind that s_i and s_o are both positive for real objects and images. Clearly, then, ***all real images formed by a single thin lens will be inverted***. The Newtonian expression for the magnification follows from Eqs. (5.19) and (5.22) and Fig. 5.24:

$$M_T = -\frac{x_i}{f} = -\frac{f}{x_o} \tag{5.26}$$

The term *magnification* is a bit of a misnomer, since the magnitude of M_T can certainly be less than 1, in which case the image is smaller than the object. We have $M_T = -1$ when the object and image distances are positive and equal, and that happens [Eq. (5.17)] only when $s_o = s_i = 2f$. This turns out to be the configuration in which the object and image are as close together as they can possibly get (i.e., a distance $4f$ apart; see Problem 5.15). Table 5.3 summarizes a number of image configurations resulting from the juxtaposition of a thin lens and a real object.

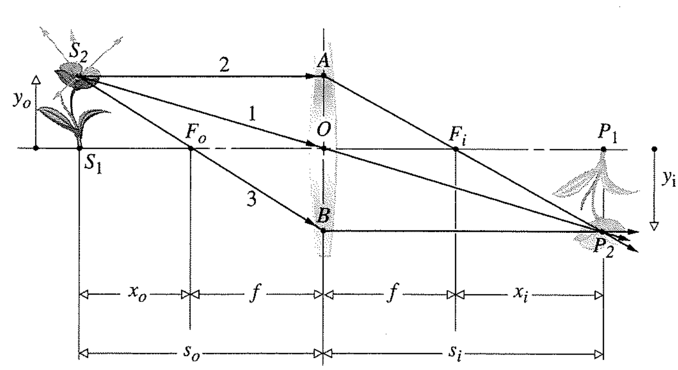


Figure 5.24 Object and image location for a thin lens.

TABLE 5.2 Meanings Associated with the Signs of Various Thin Lens and Spherical Interface Parameters

Quantity	Sign +	Sign −
s_o	Real object	Virtual object
s_i	Real image	Virtual image
f	Converging lens	Diverging lens
y_o	Erect object	Inverted object
y_i	Erect image	Inverted image
M_T	Erect image	Inverted image

EXAMPLE 5.3

A biconvex (also called a double convex) thin spherical lens has radii of 100 cm and 20.0 cm. The lens is made of glass with an index of 1.54 and is immersed in air. (a) If an object is placed 70.0 cm in front of the 100-cm surface, locate the resulting image and describe it in detail. (b) Determine the transverse magnification of the image. (c) Draw a ray diagram.

SOLUTION

(**a**) We don't have the focal length, but we do know all the physical parameters, so Eq. (5.16) comes to mind:

$$\frac{1}{f} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Leaving everything in centimeters

$$\frac{1}{f} = (1.54 - 1)\left(\frac{1}{100} - \frac{1}{-20.0}\right)$$

$$\frac{1}{f} = (0.54)\left(\frac{1}{100} + \frac{1}{20.0}\right)$$

$$\frac{1}{f} = (0.54)\frac{6}{100}$$

$$f = 30.86 \text{ cm} = 30.9 \text{ cm}$$

Now we can find the image. Since $s_o = 70.0$ cm, that's greater than $2f$—hence, even before we calculate s_i, we know that the image will be real, inverted, located between f and $2f$, and minified. To find s_i, having f we use Gauss's Equation:

$$\frac{1}{s_i} + \frac{1}{s_o} = \frac{1}{f}$$

$$\frac{1}{s_i} + \frac{1}{70.0} = \frac{1}{30.86}$$

$$\frac{1}{s_i} = \frac{1}{30.86} - \frac{1}{70.0} = 0.018\,12$$

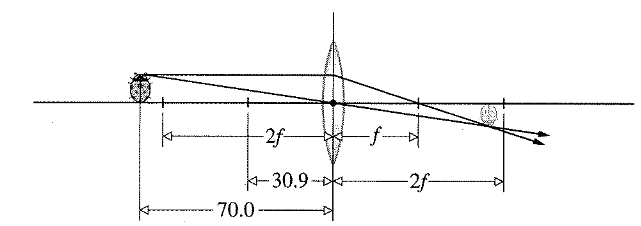
and

$$s_i = 55.19 = 55.2 \text{ cm}$$

The image is between f and $2f$ on the right of the lens. Note that $s_i > 0$, which means the image is real.
(**b**) The magnification follows from

$$M_T = -\frac{s_i}{s_o} = -\frac{55.19}{70.0} = -0.788$$

and the image is inverted ($M_T < 0$) and minified ($M_T < 1$).
(**c**) Draw the lens and mark out two focal lengths



on each side. Place the object to the left of the lens beyond $2f$. The image falls between f and $2f$.

We are now in a position to understand the entire range of behavior of a single convex or concave lens. To that end, suppose that a distant point source sends out a cone of light that is intercepted by a positive lens (Fig. 5.25). If the source is at infinity (i.e., so far away that it might just as well be infinity),

TABLE 5.3 Images of Real Objects Formed by Thin Lenses

Object	Image			
Convex				
Location	Type	Location	Orientation	Relative Size
$\infty > s_o > 2f$	Real	$f < s_i < 2f$	Inverted	Minified
$s_o = 2f$	Real	$s_i = 2f$	Inverted	Same size
$f < s_o < 2f$	Real	$\infty > s_i > 2f$	Inverted	Magnified
$s_o = f$		$\pm\infty$		
$s_o < f$	Virtual	$\lvert s_i\rvert > s_o$	Erect	Magnified
Concave				
Location	Type	Location	Orientation	Relative Size
Anywhere	Virtual	$\lvert s_i\rvert < \lvert f\rvert$, $s_o > \lvert s_i\rvert$	Erect	Minified

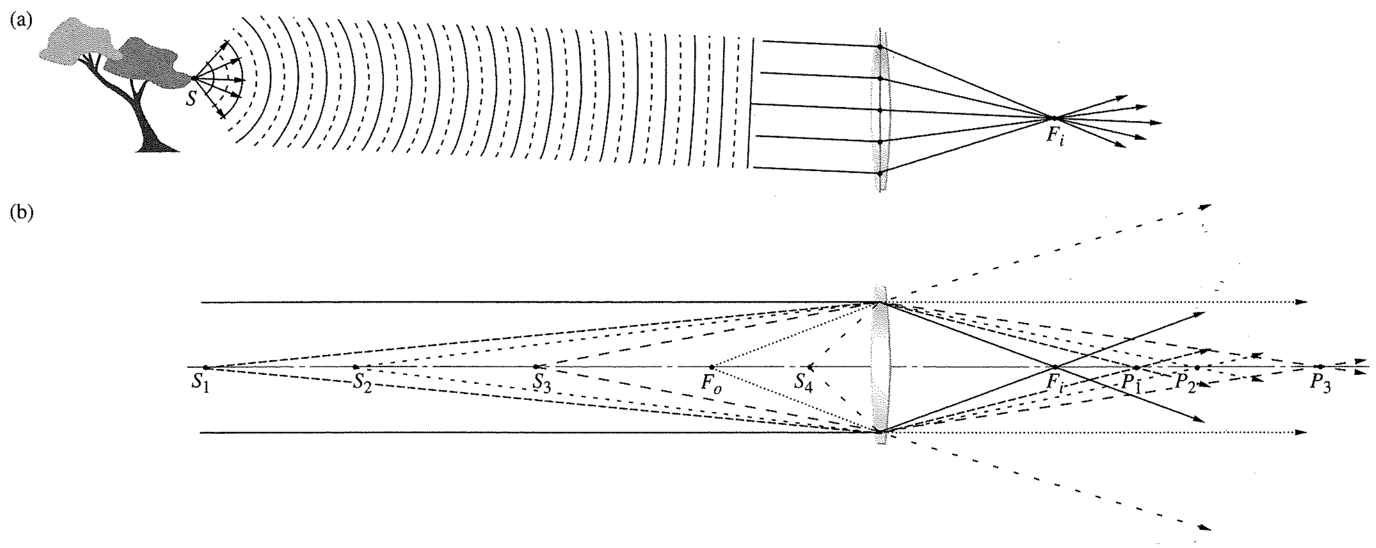


Figure 5.25 (*a*) The waves from a distant object flatten out as they expand, and the radii get larger and larger. Viewed from far away the rays from any point are essentially parallel, and the lens causes them to converge at F_i. (*b*) As a point source moves closer, the rays diverge more and the image point moves out away from the lens. The emerging rays no longer converge once the object reaches the focal point; nearer in still, they diverge.

rays coming from it entering the lens are essentially parallel (Fig. 5.25*a*) and will be brought together at the focal point F_i. If the source point-S_1 is closer (Fig. 5.25*b*), but still fairly far away, the cone of rays entering the lens is narrow, and the rays come in at shallow angles to the surface of the lens. Because the rays do not diverge greatly, the lens bends each one into convergence, and they arrive at point-P_1. As the source moves closer, the entering rays diverge more, and the resulting image point moves farther to the right. Finally, when the source point is at F_o, the rays are diverging so strongly that the lens can no longer bring them into convergence, and they emerge parallel to the central axis. Moving the source point closer results in rays that diverge so much on entering the lens that they still diverge on leaving. The image point is now virtual—*there are no real images of objects that are at or closer in than f.*

Figure 5.26 illustrates the behavior pictorially. **As the object approaches the lens, the real image moves away from it.** When the object is very far away, the image (real, inverted, and minified $M_T < 1$) is just to the right of the focal plane. As the object approaches the lens, the image (still real, inverted, and minified $M_T < 1$) moves away from the focal plane, to the right, getting larger and larger. With the object between infinity and $2f$ we have the arrangement for cameras and eyeballs, both of which require a minified, real image. By the way, it's the brain that flips the image so that you see things right-side-up.

When the object is at two focal lengths, the image (real and inverted) is now life size, that is, $M_T = 1$. This is the usual configuration of the photocopy machine.

As the object comes closer to the lens (between $2f$ and f), the image (real, inverted, and enlarged $M_T > 1$) rapidly moves to the right and continues to increase in size. This configuration corresponds to the film projector where the crucial feature is that the image is real and enlarged. To compensate for the image being inverted, the film is simply put in upside-down.

When the object arrives at a distance from the lens of precisely one focal length, the image has, in effect, moved off to infinity. (There is no image; the emerging rays are parallel.)

With the object closer in than one focal length, the image (virtual, right-side-up, and enlarged $M_T > 1$) reappears. This is the configuration of the magnifying glass. It's useful to remember that *the ray entering the lens parallel to the central axis fixes the height of the real image* (Fig. 5.27). Because that ray diverges from the central axis, the size of the image increases rapidly as the object approaches F.

EXAMPLE 5.4

Both surfaces of an equiconvex thin spherical lens have the same curvature. A 2.0-cm-tall bug is on the central axis 100 cm from the front face of the lens. The image of the bug formed on a wall is 4.0 cm tall. Given that the glass of the lens has an index of 1.50, find the radii of curvature of the surfaces.

SOLUTION

We have $y_o = 2.0$ cm, $s_o = 100$ cm, $R_1 = R_2$, $|y_i| = 4.0$ cm, and $n_l = 1.50$. We also know that the image is real, so it must be inverted and therefore $y_i = -4.0$ cm—that's crucial! To find the radii we'll need Eq. (5.16) and the focal length. We can compute f if we first determine s_i. Hence, knowing M_T,

$$M_T = \frac{y_i}{y_o} = -\frac{s_i}{s_o} = \frac{-4.0}{2.0} = -2.0$$

$$s_i = 2.0 s_o = 200 \text{ cm}$$

Continued

Using the Gaussian Lens Formula

$$\frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{100} + \frac{1}{200}$$

$$f = \frac{200}{3} = 66.67 \text{ cm}$$

The Lensmaker's Formula will give us R:

$$\frac{1}{f} = (1.50 - 1)\left(\frac{1}{R} - \frac{1}{-R}\right) = \frac{1}{2}\frac{2}{R}$$

and

$$f = R = 67 \text{ cm}$$

Note that the transformation from object to image space is not linear; all of the object space from $2f$ out to infinity, on the left of the lens, is compressed in the image space between f and $2f$, on the right of the lens. Figure 5.27 suggests that the image space is distorted, in the sense that advancing the object uniformly toward the lens has the effect of changing the image differently along and transverse to the central axis. The axial image intervals increase much more rapidly than the corresponding successive changes in the height of the image. This relative "flattening" of distant-object space is easily observable using a telescope (i.e., a long focal-length lens). You've probably seen the effect in a motion picture shot through a telephoto lens. Always staying far away, the hero vigorously runs a great distance toward the camera, but psychologically he seems to make no progress because his perceived size increases very little despite all his effort.

When an object is closer to a convex lens than one focal length (Fig. 5.26*d*) the resulting image is virtual, upright, and magnified. As listed in Table 5.3 the image is farther to the left of the lens than is the object. We can see what's happening with that virtual image in Fig. 5.28, where several objects, all of the same size, are located between the focal point-F_o and the vertex V. A number-2 ray parallel to the central axis marks the tops of all of the objects; it refracts through point-F_i and that ray,

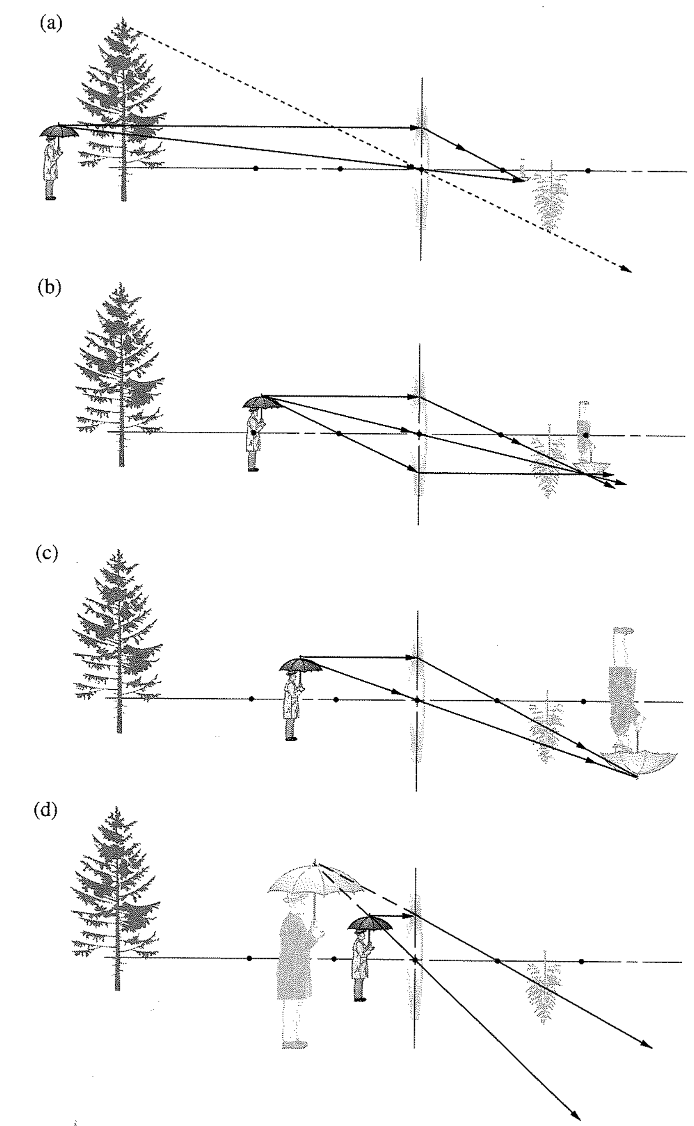


Figure 5.26 The image-forming behavior of a thin positive lens.

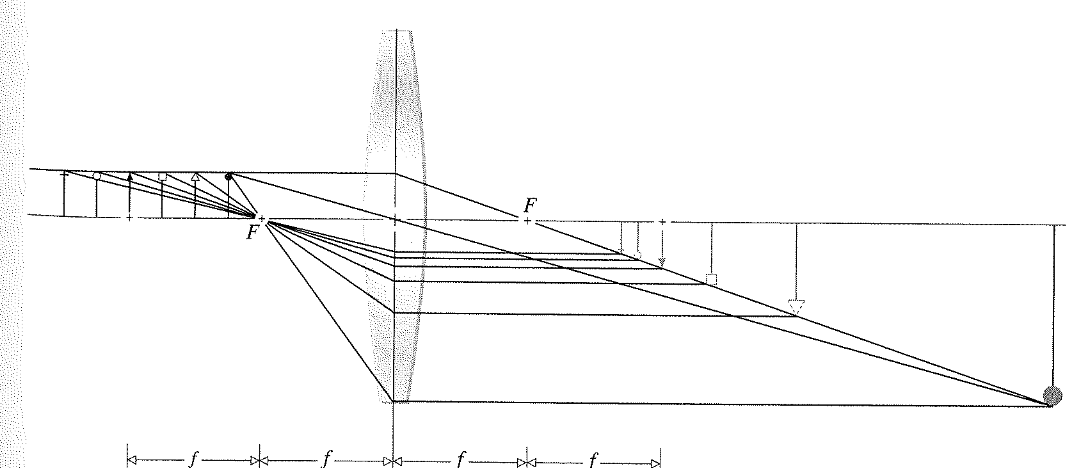


Figure 5.27 The number-2 ray entering the lens parallel to the central axis limits the image height.

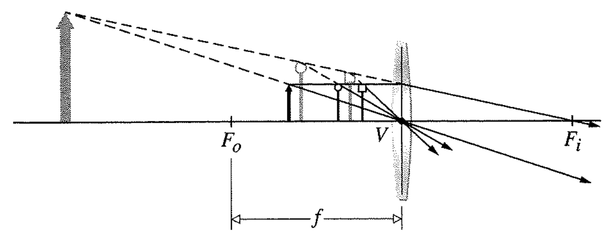


Figure 5.28 The formation of virual images by a positive lens. The closer the object comes to the lens, the closer the image approaches the lens.

projected backward, fixes the heights of each image. Notice that as the objects approach the lens the images shrink, although the magnification is still greater than 1. When the object is smack up against the lens the image is life-sized.

Longitudinal Magnification

Presumably, the image of a three-dimensional object will itself occupy a three-dimensional region of space. The optical system can apparently affect both the transverse and longitudinal dimensions of the image. The **longitudinal magnification**, M_L, which relates to the axial direction, is defined as

$$M_L \equiv \frac{dx_i}{dx_o} \tag{5.27}$$

This is the ratio of an infinitesimal axial length in the region of the image to the corresponding length in the region of the object. Differentiating Eq. (5.23) leads to

$$M_L = -\frac{f^2}{x_o^2} = -M_T^2 \tag{5.28}$$

for a thin lens in a single medium (Fig. 5.29). Evidently, $M_L < 0$, which implies that a positive dx_o corresponds to a negative dx_i and vice versa. In other words, a finger pointing toward the lens is imaged pointing away from it (Fig. 5.30).

Form the image of a window on a sheet of paper, using a simple convex lens. Assuming a lovely arboreal scene, image the distant trees on the screen. Now move the paper *away* from the lens, so that it intersects a different region of the image space. The trees will fade while the nearby window itself comes into view.

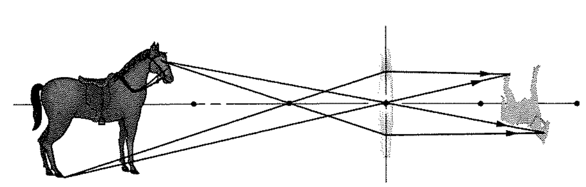


Figure 5.29 The transverse magnification is different from the longitudinal magnification.

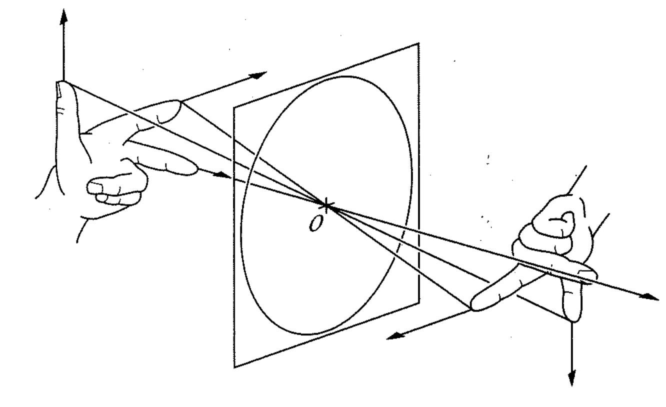


Figure 5.30 Image orientation for a thin lens.

Virtual Objects

We'll soon be studying combinations of lenses, but before we do we should consider a situation that often arises when there are several lenses in sequence. It is then possible for the rays to converge down upon a lens, as in Fig. 5.31*a*. Here the rays are symmetrically distributed about the central axis and all of them are heading toward the object focus F_o. As a result the rays exit the lens parallel to the central axis and the image is at infinity, which just means there isn't one. Because the rays converge toward the point-F_o it is customary to say that it corresponds to a virtual point object. The same is true of point-F_o in Fig. 5.31*b*, where ray-1 passing through the center of the lens makes a small angle with the axis. The rays all converge toward F_o on the focal plane and we again have a virtual point object. All of the rays leave the lens parallel to ray-1. That's an important fact to remember and we'll make use of it later.

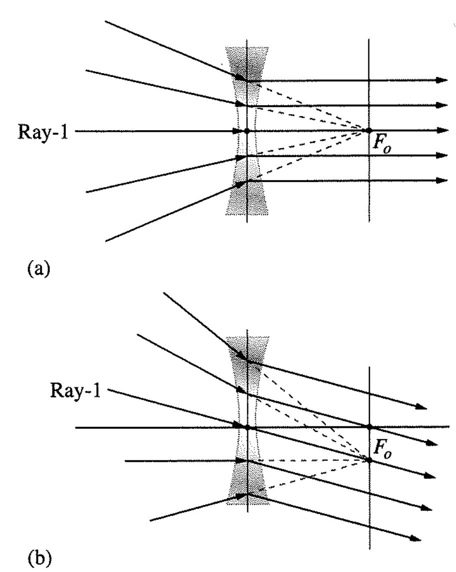


Figure 5.31 Virtual point objects for a negative lens (a) on and (b) off axis. When rays converge to the object, the object is virtual. That often happens in multi-lens systems.

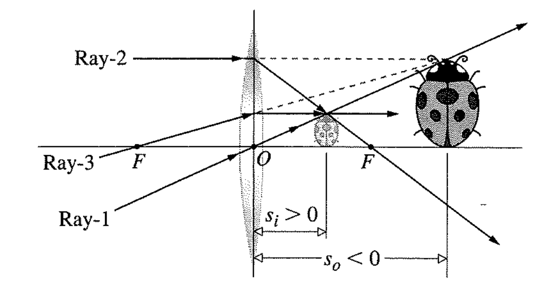


Figure 5.32 A virtual object (far right) and its real, upright image (just to the right of the lens). This can happen in a multi-lens system.

Things get a little more complicated for an extended object as in Fig. 5.32. Three converging rays enter the positive lens heading for the top of what will be the "object"—of course, nothing other than ray-1 exists at that location; the actual bug is presumably somewhere off to the left. The rays, which, before entering the lens, are directed toward the head of the object bug (far right, $s_o < 0$), are refracted by the lens and actually converge at the head of an upright, minified, real image of the bug. Notice that the object is located beyond one focal length from the lens. The lens adds convergence to the rays, which then converge to the image, which is closer to the lens but still on its right side. ***The object is virtual (s_o < 0) and the image is real (s_i > 0).*** One could place a screen at s_i and an image would appear on it. Incidentally, *when both object and image appear on the same side of a lens, one of them must be real and the other virtual.*

A somewhat similar situation exists in Fig. 5.33, where three rays again head toward the top of the "object" before entering what is this time a negative lens. That object bug being to the right of the lens ($s_o < 0$) is virtual. The rays pass through the lens, diverge, and seem to come from the inverted, minified, virtual image on the left of the lens. That is, an observer on the right looking left into the lens would pick up the three rays and projecting them back to the left would see the inverted bug image. ***The object is virtual (s_o < 0) and the image is virtual (s_i < 0).***

Notice that the virtual object in Fig. 5.33 appears beyond one focal length from the lens. If the three rays approach at greater angles they could converge toward an object that is closer

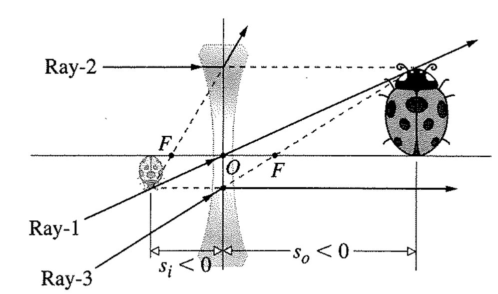


Figure 5.33 A virtual object (on the right) and its virtual, inverted image (on the left). This kind of situation can arise in a multi-lens system.

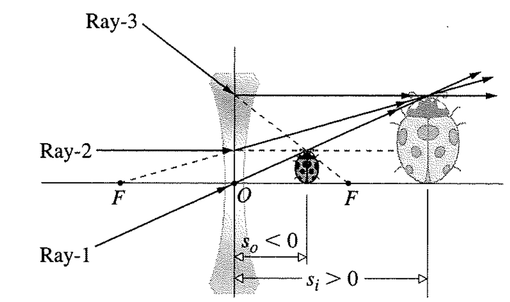


Figure 5.34 A virtual object (just to the right of the lens) and its real enlarged, upright image (far right). This can happen in a multi-lens system that causes the rays to initially converge.

to the lens (Fig. 5.34). All of the rays don't actually make it to the object and it is again ($s_o < 0$) virtual. Now when the rays are refracted by the lens they arrive at the image; the rays converge on the image, which is to the right of the lens ($s_i > 0$) and therefore real. ***The object is virtual (s_o < 0) and the image is real (s_i > 0).***

Focal-Plane Ray Tracing

Until now we've done well by simply tracing our three favorite rays, but there is another ray-tracing scheme that's well worth knowing. It's predicated on the fact that points on the focal plane of a lens are always associated with parallel columns of rays. Consequently, imagine an arbitrary ray incident on a positive lens (Fig. 5.35*a*). The ray crosses the *first focal plane* (reexamine Fig. 5.19) at point-*A*, but so far we haven't tried to pictorially determine where it goes after it refracts at point-*B*. Still, we do know that all rays from point-*A* must emerge from the lens parallel to one another. Moreover, we know that a ray

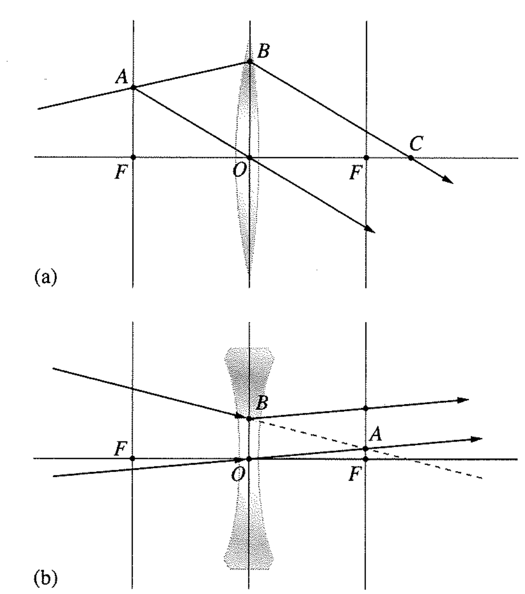


Figure 5.35 Focal-plane ray tracing. Reexamine Fig. 5.31*b*.