Contents

2 Electrostatic fields.

2.1 Basic formulas.

Given a distribution of static charges, what is the force on an arbitrary test charge at an arbitrary point?

DEFINITIONS:

Electric field $\vec{E}(\vec{r})$: force per unit of charge on test charge at point $\vec{r}$.
Charge density $\rho(\vec{r})$: amount of charge in a small volume d^3r at point $\vec{r}$ is $\rho(\vec{r})d^3r$.

BASIC RELATIONS, DIFFERENTIAL EQUATION:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho(\vec{r})$$

$$\vec{\nabla} \times \vec{E} = 0$$

FORMAL SOLUTION:

GOAL 2

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3R\rho(\vec{R})\frac{\vec{r}-\vec{R}}{|\vec{r}-\vec{R}|^3}$$

Integral forms:

$$\oint d^2S \cdot \vec{E} = \frac{1}{\epsilon_0}Q_{enclosed}$$

$$\oint d\vec{L} \cdot \vec{E} = 0$$

Special cases:

Field of point charges

$$\rho(\vec{r}) = \sum_i Q_i\delta(\vec{r}-\vec{R}_i)$$

$$\vec{E}(\vec{r}) = \sum_i \frac{Q_i}{4\pi\epsilon_0}\frac{\vec{r}-\vec{R}_i}{|\vec{r}-\vec{R}_i|^3}$$

Field of line charges

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int dR\lambda(\vec{R})\frac{\vec{r}-\vec{R}}{|\vec{r}-\vec{R}|^3}$$

Field of surface charges

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^2R\sigma(\vec{R})\frac{\vec{r}-\vec{R}}{|\vec{r}-\vec{R}|^3}$$

Definition:

Potential $V(\vec{r})$ is potential energy per unit charge of test charge at point $\vec{r}$. Possible because Coulomb force is conservative.

$$\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$$

Basic relations, differential equation:

$$\Delta V(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r})$$

Formal solution:

GOAL 2

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3R \rho(\vec{R}) \frac{1}{|\vec{r} - \vec{R}|}$$

2.2 Solutions in special cases.

Point charge , infinite straight line , infinite plane, infinite cylinder, etc.

GOAL 1

High symmetry: use Gauss (integral form)!

Potential far away from localized charge density:

GOAL 3

Multipole expansion:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_l \frac{1}{r^{l+1}} M_l(\hat{r})$$

$$M_l(\hat{r}) = \int d^3R R^l P_l(\hat{r} \cdot \hat{R}) \rho(\vec{R})$$

Monopole: $Q = \int d^3R \rho(\vec{R})$
Dipole: $p_i = \int d^3R R_i \rho(\vec{R})$
Quadrupole: $Q_{ij} = \int d^3R (3R_i R_j - R^2 \delta_{ij}) \rho(\vec{R})$

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left(\frac{1}{r} Q + \frac{1}{r^2} \sum_i \hat{r}_i p_i + \frac{1}{r^3} \frac{1}{2} \sum_{ij} \hat{r}_i \hat{r}_j Q_{ij} + \cdots \right)$$

Regions without charge, use separation of variables and boundary conditions.

Cartesian coordinates.

$$\Delta V = 0$$

If $V(\vec{r}) = X(x)Y(y)Z(z)$ then

$$\frac{d^2X}{dx^2} = C_x X$$

$$\frac{d^2Y}{dy^2} = C_y Y$$

$$\frac{d^2Z}{dz^2} = C_z Z$$

with $C_x + C_y + C_z = 0$.

GOAL 1

GOAL 2

General solution: sum over all allowed values of C_x, C_y.

Spherical coordinates.

GOAL 1

GOAL 2

In this case the angles θ, ϕ are restricted and the corresponding solutions have integral quantum number.
General solution:

$$V(\vec{r}) = \sum_{lm} \left(A_{lm} r^l + B_{lm} r^{-l-1} \right) Y_l^m(\theta, \phi)$$

Spherical coordinates, cylindrical symmetry:

$$V(\vec{r}) = \sum_{l} \left(A_l r^l + B_l r^{-l-1} \right) P_l(\cos(\theta))$$

Boundary condition at conducting surface:

$$V(\vec{r}) = V_0$$

$$\vec{E}(\vec{r}) = \frac{\sigma}{\epsilon_0} \hat{n}$$

Mirror images.

Field of point charge Q in front of conducting plane is equal to field of Q and field of -Q at mirror image.

2.3 Charges bound to material.

Dipole moment: $\vec{p} = q\vec{d}$
Torque on dipole: $\tau = \vec{p} \times \vec{E}$
Energy dipole: $U = -\vec{p} \cdot \vec{E}$
Polarization density: $\vec{P} = n\vec{p}$ for n units per unit volume

Bound charges and polarization density.

GOAL 5

$$\sigma_b = \vec{P} \cdot \hat{n}$$

$$\vec{\nabla} \cdot \vec{P} = -\rho_b$$

Displacement field.

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_{free}$$

Linear medium.

$$\vec{P} = \chi \epsilon_0 \vec{E}$$

$$\vec{D} = \epsilon \vec{E}$$

$$\epsilon = \epsilon_r \epsilon_0$$

$$\epsilon_r = 1 + \chi$$

Boundary conditions at dielectric boundary.

GOAL 4

GOAL 5

$$\vec{D}_1 \cdot \hat{n} = \vec{D}_2 \cdot \hat{n}$$

$$\vec{E}_1 \times \hat{n} = \vec{E}_2 \times \hat{n}$$

2.4 Additional information.

Energy of charge distribution.

$$U = \frac{1}{2} \int d^3 R_1 d^3 R_2 \frac{\rho(\vec{R}_1)\rho(\vec{R}_2)}{|\vec{R}_1 - \vec{R}_2|}$$

$$U = \frac{1}{2} \epsilon_0 \int d^3 r \vec{E}^2(\vec{r})$$

Capacitance.

GOAL 6

GOAL 7

$$C = \frac{Q}{\Delta V}$$

$$U = \frac{1}{2} C V^2$$

Parallel plate: $C = \epsilon \frac{S}{d}$

3 Magnetostatics.

Given a static current distribution, what is the force on a magnetic pole at arbitrary points?

Static current density distribution $\vec{J}(\vec{R})$ in Coulomb per second per unit area does not change with time.

3.1 Basic current definitions.

Conductivity σ from $\vec{J} = \sigma \vec{E}$
Resistance $R = \frac{V}{I}$ depends on geometry.
$\sigma = -e\mu n_e$ in terms of mobility μ and density n.
Dissipated power $P = IV$
Continuity equation $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

3.2 Fields due to currents.

Basic relations, differential equation:

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Formal solution:

$$\vec{E}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3R \vec{J}(\vec{R}) \times \frac{\vec{r} - \vec{R}}{|\vec{r} - \vec{R}|^3}$$

Integral forms:

$$\oint d^2S \cdot \vec{B} = 0$$

$$\oint d\vec{L} \cdot \vec{B} = \mu_0 I_{enclosed}$$

Special cases:

Field of current loop

GOAL 9

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3} \left(3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}\right)$$

$$\vec{m} = I\vec{A}$$

3.3 Solutions in special cases.

Infinite straight line , infinite coil, etc.

GOAL 8

High symmetry: use Ampere (integral form)!

Potential far away from localized charge density:

GOAL 10

Multipole expansion. More complicated. No monopoles. Lowest order:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left(3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right)$$

$$\vec{m} = \frac{1}{2}\int d^3R\vec{R}\times\vec{J}(\vec{r})$$

Intrinsic dipoles due to spin of particle.

3.4 Forces.

$$\vec{F} = \int d^3r\left(\rho(\vec{r})\vec{E}(\vec{r}) + \vec{J}(\vec{r})\times\vec{B}(\vec{r})\right)$$

Torque on dipole

$$\tau = \vec{m}\times\vec{B}$$

3.5 Currents bound to material.

GOAL 11

Magnetization density $\vec{M} = n\vec{m}$

Bound currents and magnetization density.

GOAL 12

$$\vec{\nabla}\cdot\vec{M} = \vec{J}_b$$

$$\vec{M}\times\hat{n} = \vec{K}_b$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free}$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M})$$

Linear medium.

$$\vec{M} = \chi \vec{H}$$

$$\vec{B} = \mu \vec{H}$$

$$\mu = \mu_r \mu_0$$

$$\mu_r = 1 + \chi$$

Boundary conditions at magnetic boundary.

GOAL 11

GOAL 12

$$\vec{B}_1 \cdot \hat{n} = \vec{B}_2 \cdot \hat{n}$$

$$\vec{H}_1 \times \hat{n} = \vec{H}_2 \times \hat{n}$$