Problem 1 An insulating vertical wood pole 1.2 m tall is topped by a hoop of radius a = 5 cm, which lies in a vertical plane. A thunderstorm overhead causes the ring to have a linear charge density

$$\lambda(s) = \Lambda \cos\left(\frac{s}{a}\right)$$

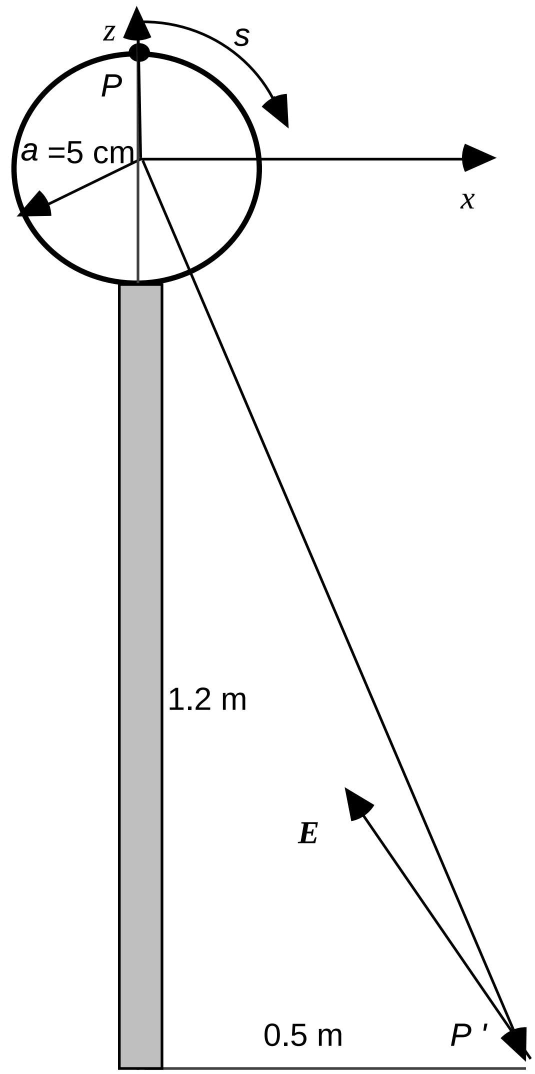
where s is the distance along the circumference of the ring from the point P at the top of the ring.

Find the electric field $\boldsymbol{E}$ due to the ring, at a point P' which lies in the plane of the ring at a horizontal displacement of 0.5 m from the base of the pole. Give its magnitude and direction, or components in a coordinate system which you may define.

choose axes z vertical, x horizontal in plane, origin at center of ring

You may make a reasonable approximation, valid to a few percent, to simplify the computation.

drawing not to scale



distance large compared to size of charge distribution use multipole expansion **GOAL #3**

total charge = $\int_0^{2\pi a} ds\ \lambda(s) = 0$ so look at dipole moment; new variable $\theta = s/a$, $ds = a\,d\theta$

$$\boldsymbol{p} = \int_0^{2\pi a} ds\ \lambda(s)\, \boldsymbol{r}(s) = \int_0^{2\pi a} ds\ \lambda(s)\left[\hat{\boldsymbol{x}}\ a \sin\left(\frac{s}{a}\right) + \hat{\boldsymbol{z}}\ a \cos\left(\frac{s}{a}\right)\right] = \Lambda a^2 \int_0^{2\pi} d\theta \cos\theta \left[\hat{\boldsymbol{x}} \sin\theta + \hat{\boldsymbol{z}} \cos\theta\right]$$

$$\boldsymbol{p} = \Lambda a^2 \pi\ \hat{\boldsymbol{z}}\ ,\ \boldsymbol{E}(\boldsymbol{r}') = \frac{3(\boldsymbol{p}\bullet\boldsymbol{r}')\boldsymbol{r}' - r'^2\boldsymbol{p}}{4\pi\varepsilon_0 r'^5}\ ,\ r' = \sqrt{(1.2\text{m})^2 + (0.5\text{m})^2} = 1.3\text{m}$$

$$\boldsymbol{p} = \frac{\Lambda(0.05\text{m})^2}{4\pi\varepsilon_0^5(1.3\text{m})^5}\left(3(-1.2\text{m})(0.5\text{m}\hat{\boldsymbol{x}} - 1.2\text{m}\hat{\boldsymbol{z}}) - (1.3\text{m})^2\hat{\boldsymbol{z}}\right) = \frac{\Lambda(0.05\text{m})^2}{4\pi\varepsilon_0^5(1.3\text{m})^5}\left(-1.8\text{m}^2\hat{\boldsymbol{x}} + 2.63\text{m}^2\hat{\boldsymbol{z}}\right)$$

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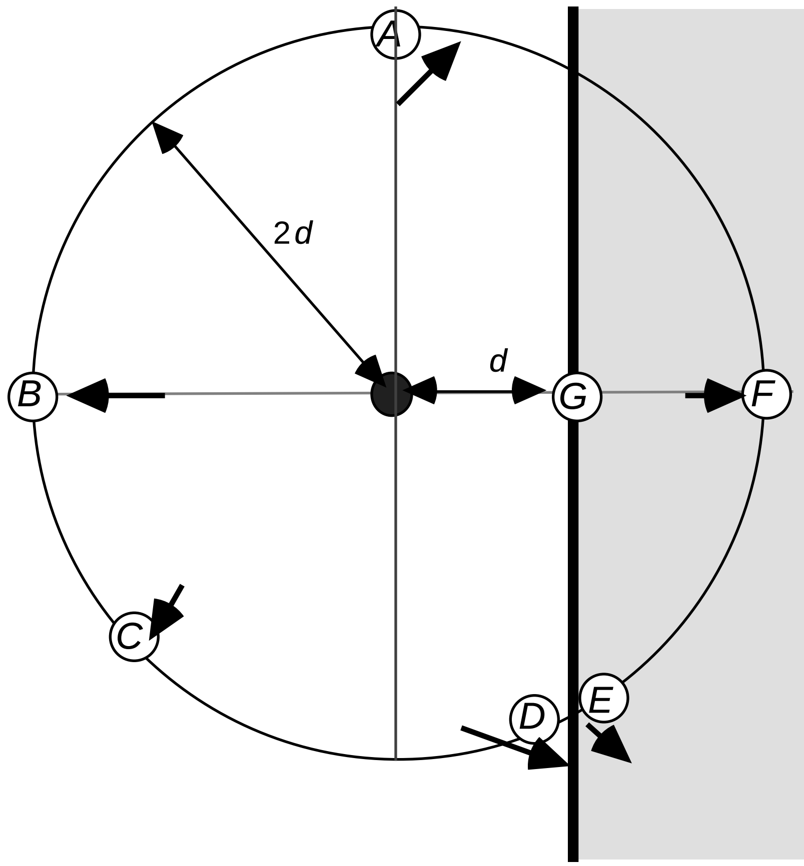
Problem 2 A long thin straight wire carries a linear charge density λ coulombs per meter. It lies parallel to the surface of a dielectric, at a distance of d meters. The dielectric's permittivity is $\varepsilon = 1.7\,\varepsilon_o$. Assume $\lambda > 0$. **FIELD OF LINE: GOAL#1 or #2**

a. Points *A, B, C, D, E* and *F* all lie at a distance of $2d$ from the wire. Draw arrows indicating the approximate direction and relative magnitude of the electric field ***E*** at each point.



b. Find the magnitude of the electric field at points *B* and *F*. Express your answers in terms of λ, d, and ε_o.

c. Find the induced charge density σ at point *G* on the surface of the dielectric closest to the wire.

d. What is the charge density at point *F* ?

e. Find the force per unit length on the wire. Give magnitude and direction.

outside: see image line at F, density λ', $E\,(B) = \dfrac{\lambda}{2\pi\varepsilon_o(2d)} + \dfrac{\lambda'}{2\pi\varepsilon_o(4d)}$, $E_{out}(G) = \dfrac{\lambda-\lambda'}{2\pi\varepsilon_o(d)}$

inside: see image line at wire, density λ'', $E(F) = \dfrac{\lambda''}{2\pi\varepsilon(2d)}$, $E_{in}(G) = \dfrac{\lambda''}{2\pi\varepsilon(d)}$ **GOAL # 4**

Find λ', λ'' by matching D at G: $\varepsilon_o E_{out}(G) = D_{out}(G) = D_{in}(G) = \varepsilon E_{in}(G)$ $\quad \lambda-\lambda' = \lambda''$

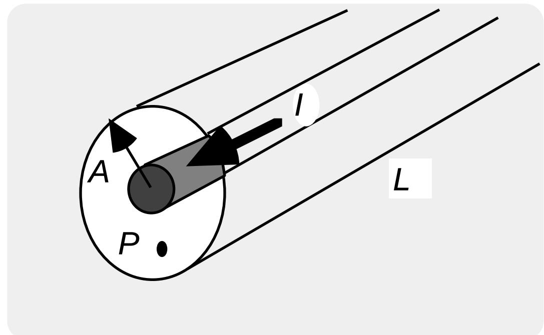
and $E_{||}$ at D,E : $\dfrac{\lambda+\lambda'}{\varepsilon_o} = \dfrac{\lambda''}{\varepsilon}$; combine $\quad \lambda'' = \dfrac{\varepsilon(\lambda+\lambda')}{\varepsilon_o} = \lambda-\lambda' \quad \lambda' = \dfrac{\varepsilon_0-\varepsilon}{\varepsilon_o+\varepsilon}\lambda$, $\lambda'' = \dfrac{\varepsilon_o}{\varepsilon_o+\varepsilon}\lambda$

surface charge at $G = \varepsilon_o(E_{in}-E_{out}) = \dfrac{\lambda''-\lambda+\lambda'}{2\pi d} = \dfrac{\lambda}{2\pi d}\dfrac{\varepsilon_o-2\varepsilon}{\varepsilon_o+\varepsilon}$, $\quad \bullet \boldsymbol{E}_{in} = 0 \quad \rho(F) = 0$. **GOAL #5**

Force on $dQ = \lambda dL$ due to image: $dF = E\,dQ = \dfrac{\lambda'}{4\pi\varepsilon_o(2d)^2}\lambda\,dL$, $\dfrac{dF}{dL} = \dfrac{\lambda\lambda'}{16\pi\varepsilon_o d^2}$ to right **GL #7**

Problem 3

A long straight conducting wire of radius *a* is centered in a cylindrical hole of radius *A* . The wire carries a current *I* , and its length *L* is much larger than *A*.



The hole is surrounded by an infinite linear magnetic material of permeability $\mu = 10\,\mu_o$.

a. In what direction is the magnetic field in the hole at point *P*, a distance $r < A$ from the center of the wire? Answer in words and by drawing an arrow.

azimuthal, follows right-hand rule counterclockwise

Show that its magnitude is $B(r) = \dfrac{\mu_o\, I}{2\pi\, r}$ (assume $r > a$.).

solution by symmetry : **GOAL #8**

apply Ampere's law to a circular path at radius *r*: $\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o I_{enclosed}$

symmetry $\boldsymbol{B}$ azimuthal, along d$\boldsymbol{l}$, independent of azimuthal angle $\oint \boldsymbol{B} \bullet d\boldsymbol{l} = B \oint dl = 2\pi r B$

$r > a$ enclosed current = I, solve for $B = \dfrac{\mu_o\, I}{2\pi\, r}$

(can also be solved by superposition, using Biot-Savart formula **GOAL #9**)

b. Find the magnetic field in the material at a distance $r > A$ from the wire's center. Give direction and magnitude. **GOAL #11**

The symmetry arguments give the same configuration for $\boldsymbol{B}$ and $\boldsymbol{H}$: azimuthal, independent of φ

Find H from integral form of $\times \boldsymbol{H} = \boldsymbol{J}_{free}$: $\oint \boldsymbol{H} \bullet d\boldsymbol{l} = I_{free} = I = 2\pi r H$, then $\boldsymbol{B} = \mu \boldsymbol{H}$ in material,

$$B(r) = \frac{\mu\, I}{2\pi\, r}$$

c. Find the effective current distribution in the magnetic material. What is the total effective current on the inner surface? In what direction does it flow? Are there any other effective currents in the material? **GOAL #12**

The extra $\boldsymbol{B}$ is due to an effective surface current, which must be $(\mu - \mu_o)/\mu_o$ times the original current: $I_{effective} = (\mu - \mu_o) I / \mu_o$.

The effective current is parallel to I and distributed uniformly around the inner circumference of the material, an effective surface current density $K_{eff} = I_{eff}/2\pi A$.

There are no other effective currents, since this alone is sufficient to give the desired B.

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Problem 4 In a simple hypothetical model, the earth's observed magnetic field might be thought to be caused by a small spherical core of fully magnetized iron at the center of the earth. If

- the magnetic field inside fully magnetized iron is about 2 Tesla, and
- the largest field observed at the earth's surface is about 0.5×10^{-4} Tesla, and
- the radius of the earth is about 6×10^{6} meters, and
- none of the material outside the core were magnetized,

what would be the radius of the magnetized core? **GOAL #10**

Which of the assumptions seems worst?

If the core is small, then we can use the dipole approximation to find the field far away from it. We need the dipole moment of the core, which is the total of the magnetization

$$\boldsymbol{m} = \int d\mathit{volume}\, \boldsymbol{M} \ .$$

We can find the magnetization $\boldsymbol{M}$ by using $\boldsymbol{B} = \mu_o(\boldsymbol{H} + \boldsymbol{M})$.

Since the model says nothing about free currents in the core, we can assume $\boldsymbol{H}$ is zero there, so

$$\boldsymbol{M} = \boldsymbol{B}/\mu_0 \qquad \boldsymbol{m} = \boldsymbol{M} \times \mathit{volume} = \frac{4\pi R_{\mathrm{core}}^3}{3\mu_o} \boldsymbol{B}_{\mathrm{core}} \ .$$

The magnetic field far from the core is $\boldsymbol{B}_{\mathrm{dipole}}(\boldsymbol{r}) = \frac{\mu_o}{4\pi r^3}\left(3(\boldsymbol{m} \bullet \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}\right)$

which is largest when $\boldsymbol{r}$ is in the same direction as $\boldsymbol{m}$. Its maximum value is

$$B_{\max}(r) = \frac{2\mu_o|m|}{4\pi r^3} = \frac{2R_{\mathrm{core}}^3}{3r^3}|\boldsymbol{B}_{\mathrm{core}}| \text{ from which}$$

$$R_{\mathrm{core}} = r\sqrt[3]{\frac{3B_{\max}(r)}{2|\boldsymbol{B}_{\mathrm{core}}|}} = 6 \times 10^6 \text{ m} \times \left(\frac{3 \times 0.5 \times 10^{-4} \text{ T}}{2 \times 2\text{T}}\right)^{1/3} \approx 2 \times 10^5 \text{ m}$$

Since most of the earth's bulk is thought to be ferrous material it seems unlikely that the magnetization would be limited to so small a volume.

An alternative interpretation of this observation is that the earth's magnetization is a small fraction of a percent of its maximum possible value.

END OF SOLUTION