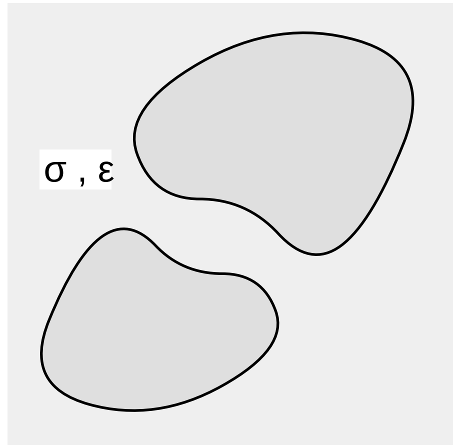
Resistance and Capacitance

Griffiths 7.3 Two ideal electrodes of arbitrary shape are immersed in a liquid of dielectric permittivity ε and conductivity σ.



a. Show that the capacitance C and resistance R are related by $R = \frac{\varepsilon}{\sigma C}$.

assume charges $\pm Q_{\text{free}}$ on electrodes

$$R = \frac{V}{I}\ , C = \frac{Q_{\text{free}}}{V} \qquad V = \frac{Q_{\text{free}}}{C} \qquad R = \frac{Q_{\text{free}}}{I\ C}\ ,$$

compute current $I = \oint \boldsymbol{J} \cdot d(\boldsymbol{area})$ over surface containing one electrode,

but $\boldsymbol{J} = \sigma \boldsymbol{E}$, so $\quad I = \sigma \oint \boldsymbol{E} \cdot d(\boldsymbol{area}) = \sigma \frac{Q_{\text{enclosed}}}{\varepsilon_o}$ by Gauss' law.

For Q_{free} , need $\boldsymbol{D} = \varepsilon \boldsymbol{E}$

Similarly, $Q_{\text{free}} = \oint \boldsymbol{D} \cdot d(\boldsymbol{area}) = \varepsilon \oint \boldsymbol{E} \cdot d\,(\boldsymbol{area}) \qquad I = \frac{\sigma Q_{\text{free}}}{\varepsilon}$

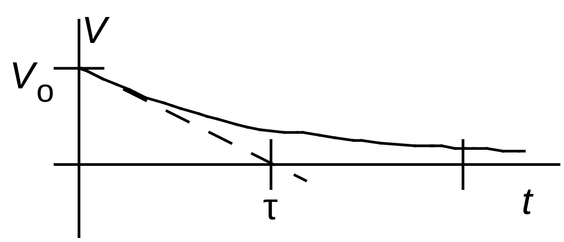
so $R = \frac{Q_{\text{free}}}{I\ C} = \frac{\varepsilon}{\sigma C}$ as claimed

b. At time $t = 0$, a battery is used to establish a potential difference V_0 between the conductors. After the battery is disconnected, the charge will gradually leak off. Show that $V(t) = V_0\, e^{-t/\tau}$, and find the "time constant" τ **.**

$$\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt} = \frac{-I}{C} = \frac{-V}{RC} \rightarrow \frac{dV}{V} = -\frac{dt}{RC} \rightarrow \ln\frac{V}{V_0} = \frac{-t}{RC}$$

time constant

$$\tau = RC = \frac{\varepsilon}{\sigma}$$



This is a slight generalization of Griffiths' problem, taken from his 2nd ed.