ROTATIONS IN SPACE

1. INTRODUCTION

In Cartesian coordinates, the natural orthonormal basis is $\{\vec{\imath},\vec{\jmath},\vec{k}\}$, where $\vec{\imath} \equiv \hat{x}$, $\vec{\jmath} \equiv \hat{y}$, $\vec{k} \equiv \hat{z}$ denote the unit vectors in the x, y, z directions, respectively. The position vector from the origin to the point (x,y,z) takes the form

$$\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$$

Note that $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ are constant.

A moving object has a position vector given by

$$\vec{r}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath} + z(t)\,\vec{k}$$

Its velocity $\vec{v}$ and acceleration $\vec{a}$ are obtained by differentiation, resulting in

$$\vec{v} = \dot{\vec{r}} = \dot{x}\,\vec{\imath} + \dot{y}\,\vec{\jmath} + \dot{z}\,\vec{k}$$
$$\vec{a} = \ddot{\vec{r}} = \ddot{x}\,\vec{\imath} + \ddot{y}\,\vec{\jmath} + \ddot{z}\,\vec{k}$$

where dots denote differentiation with respect to t.

As in the plane, if we wish to describe things as seen from some point (p, q, s) other than the origin, all we have to do is replace $\vec{r}$ by

$$\vec{r}_{rel} = \vec{r} - \vec{R}$$

where

$$\vec{R} = p\,\vec{\imath} + q\,\vec{\jmath} + s\,\vec{k}$$

2. SPHERICAL COORDINATES

In spherical coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

the natural orthonormal basis is $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$, where

$$\hat{r} = \sin\theta\cos\phi\,\vec{\imath} + \sin\theta\sin\phi\,\vec{\jmath} + \cos\theta\,\vec{k}$$
$$\hat{\theta} = \cos\theta\cos\phi\,\vec{\imath} + \cos\theta\sin\phi\,\vec{\jmath} - \sin\theta\,\vec{k}$$
$$\hat{\phi} = -\sin\phi\,\vec{\imath} + \cos\phi\,\vec{\jmath}$$

Again, this is a basis everywhere *except* at the origin, since neither θ nor ϕ are defined there.

Consider an observer located at the point (R,Θ,Φ), *not* at the origin, whose natural basis is just $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$. With respect to this observer, a moving object therefore has a *relative* position vector of the form

$$\vec{r}_{rel}(t) = Z(t)\,\hat{r} - Y(t)\,\hat{\theta} + X(t)\,\hat{\phi}$$

for some functions X, Y, Z.

We will normally take our observer to be on the surface of the Earth, which we will approximate as a sphere. Roughly speaking, Θ and Φ give the latitude and longitude of the observer, respectively; R is the radius of the Earth.

Note that the equator corresponds to $\theta = \frac{\pi}{2}$, and that θ *decreases* as you approach the North Pole. The functions (X,Y,Z) define *Cartesian* coordinates for the observer: Z is altitude, Y is distance to the *north*, and X is distance to the *east*; this explains the peculiar conventions in defining X, Y, Z.

The "true" position is given by

$$\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{R}} + \vec{\boldsymbol{r}}_{rel}(t)$$

where

$$\vec{\boldsymbol{R}} = R\,\hat{\boldsymbol{r}}$$

3. ROTATING FRAME

An observer "standing still" on the surface of the Earth is really a rotating observer whose position is given by

$$\begin{aligned} r &= R = \text{constant} \\ \theta &= \Theta = \text{constant} \\ \phi &= \Phi = \Omega t \end{aligned}$$

Observers in this frame perceive $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ to be constant. (The sun "rises" and "sets"!) They will therefore compute the relative velocity and acceleration of a moving object with (relative) position vector $\vec{\boldsymbol{r}}_{rel}$ by taking derivatives of the *coefficients*:

$$\begin{aligned} \vec{\boldsymbol{v}}_{rel}(t) &= \dot{Z}\,\hat{\boldsymbol{r}} - \dot{Y}\,\hat{\boldsymbol{\theta}} + \dot{X}\,\hat{\boldsymbol{\phi}} \\ \vec{\boldsymbol{a}}_{rel}(t) &= \ddot{Z}\,\hat{\boldsymbol{r}} - \ddot{Y}\,\hat{\boldsymbol{\theta}} + \ddot{X}\,\hat{\boldsymbol{\phi}} \end{aligned}$$

The basis vectors now take the form

$$\begin{aligned} \hat{\boldsymbol{r}} &= \sin\theta\cos(\Omega t)\,\vec{\boldsymbol{\imath}} + \sin\theta\sin(\Omega t)\,\vec{\boldsymbol{\jmath}} + \cos\theta\,\vec{\boldsymbol{k}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos(\Omega t)\,\vec{\boldsymbol{\imath}} + \cos\theta\sin(\Omega t)\,\vec{\boldsymbol{\jmath}} - \sin\theta\,\vec{\boldsymbol{k}} \\ \hat{\boldsymbol{\phi}} &= -\sin(\Omega t)\,\vec{\boldsymbol{\imath}} + \cos(\Omega t)\,\vec{\boldsymbol{\jmath}} \end{aligned}$$

so that

$$\begin{aligned} \dot{\hat{\boldsymbol{r}}} &= -\Omega\sin\theta\sin(\Omega t)\,\vec{\boldsymbol{\imath}} + \Omega\sin\theta\cos(\Omega t)\,\vec{\boldsymbol{\jmath}} \\ \dot{\hat{\boldsymbol{\theta}}} &= -\Omega\cos\theta\sin(\Omega t)\,\vec{\boldsymbol{\imath}} + \Omega\cos\theta\cos(\Omega t)\,\vec{\boldsymbol{\jmath}} \\ \dot{\hat{\boldsymbol{\phi}}} &= -\Omega\cos(\Omega t)\,\vec{\boldsymbol{\imath}} - \Omega\sin(\Omega t)\,\vec{\boldsymbol{\jmath}} \end{aligned}$$

Comparing these equations with the preceding ones, we see that

$$\begin{aligned} \dot{\hat{\boldsymbol{r}}} &= \Omega\sin\theta\,\hat{\boldsymbol{\phi}} = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{r}} \\ \dot{\hat{\boldsymbol{\theta}}} &= \Omega\cos\theta\,\hat{\boldsymbol{\phi}} = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\theta}} \\ \dot{\hat{\boldsymbol{\phi}}} &= -\Omega\sin\theta\,\hat{\boldsymbol{r}} - \Omega\cos\theta\,\hat{\boldsymbol{\theta}} = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\phi}} \end{aligned}$$

where we have introduced the angular velocity

$$\vec{\boldsymbol{\omega}} = \Omega\,\vec{\boldsymbol{k}} = \Omega\left(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Thus, just as in the 2-dimensional case, for *any* relative vector of the form

$$\vec{\boldsymbol{F}}(t) = f(t)\,\hat{\boldsymbol{r}}(t) + g(t)\,\hat{\boldsymbol{\theta}}(t) + h(t)\,\hat{\boldsymbol{\phi}}(t)$$

we have

$$\begin{aligned}
\dot{\vec{\boldsymbol{F}}} &= \left(\dot f\,\hat{\boldsymbol{r}} + \dot g\,\hat{\boldsymbol{\theta}} + \dot h\,\hat{\boldsymbol{\phi}}\right) + \left(f\,\dot{\hat{\boldsymbol{r}}} + g\,\dot{\hat{\boldsymbol{\theta}}} + h\,\dot{\hat{\boldsymbol{\phi}}}\right)\\
&= \left(\dot f\,\hat{\boldsymbol{r}} + \dot g\,\hat{\boldsymbol{\theta}} + \dot h\,\hat{\boldsymbol{\phi}}\right) + \left(f\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{r}} + g\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\theta}} + h\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\phi}}\right)\\
&= \left(\dot f\,\hat{\boldsymbol{r}} + \dot g\,\hat{\boldsymbol{\theta}} + \dot h\,\hat{\boldsymbol{\phi}}\right) + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{F}}
\end{aligned}$$

As before, the first term is the "naive" derivative of $\vec{\boldsymbol{F}}$; this "naive" differentiation is precisely what was used to obtain $\vec{\boldsymbol{v}}_{rel}$ and then $\vec{\boldsymbol{a}}_{rel}$ starting from $\vec{\boldsymbol{r}}_{rel}$.

We are finally ready to compare the relative and "true" velocities and accelerations. Differentiating

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{R}} + \vec{\boldsymbol{r}}_{rel}$$

we obtain

$$\begin{aligned}
\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}} &= \dot{\vec{\boldsymbol{R}}} + \dot{\vec{\boldsymbol{r}}}_{rel}\\
&= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{R}} + (\vec{\boldsymbol{v}}_{rel} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{rel})
\end{aligned}$$

Further differentiation yields

$$\begin{aligned}
\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} &= \vec{\boldsymbol{\omega}} \times \dot{\vec{\boldsymbol{R}}} + \left(\dot{\vec{\boldsymbol{v}}}_{rel} + \vec{\boldsymbol{\omega}} \times \dot{\vec{\boldsymbol{r}}}_{rel}\right)\\
&= \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{R}}) + (\vec{\boldsymbol{a}}_{rel} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{rel}) + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{v}}_{rel} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{rel})\\
&= \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{R}}) + \vec{\boldsymbol{a}}_{rel} + 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{rel} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{rel})
\end{aligned}$$

Rewriting these expressions slightly, we obtain the following equations for the relative velocity and acceleration:

$$\begin{aligned}
\vec{\boldsymbol{v}}_{rel} &= \vec{\boldsymbol{v}} - \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}\\
\vec{\boldsymbol{a}}_{rel} &= \vec{\boldsymbol{a}} - 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{rel} - \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})
\end{aligned}$$

As before, the *effective* acceleration $\vec{\boldsymbol{a}}_{rel}$ therefore consists of 3 parts: the "true" acceleration $\vec{\boldsymbol{a}}$, the *centrifugal* acceleration $-\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$ and the *Coriolis* acceleration $-2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{rel}$.

On the surface of the Earth, we have

$$\vec{\boldsymbol{r}} \approx \vec{\boldsymbol{R}}$$

so that the centrifugal acceleration can be approximated as $-\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{R}})$. In analogy with the planar case, the centrifugal acceleration always points away from the axis of the Earth's rotation, and is strongest at the equator. This acceleration is perceived as a small (less than 1%) correction to the acceleration due to gravity: Straight down, as defined by a plumb bob, does *not* point towards the center of the Earth! [1] For a more detailed discussion, see pages 390–391 of Marion and Thornton.

Finally, *for motion parallel to the surface of the Earth*, the Coriolis acceleration always points to the *right* of the direction of motion $\vec{\boldsymbol{v}}_{rel}$ in the Northern Hemisphere (and to the *left* in the Southern Hemisphere).

[1] Straight down is, however, perpendicular to the surface of the Earth: The centrifugal force has deformed the Earth's surface, resulting in an equatorial radius which is slightly (just over 20 km) greater than the polar radius.