solutions to
TRANSLATING TENSORS
Worksheet 3

by Philip J. Siemens

TOPICS page

A. CHECKING PREVIOUS EXERCISES

We can use formula C.7 in the chapter on Rigid Bodies to compare the tensors we calculated in the worksheet *Inertial Integrals*, in the center of mass and another coordinate system. (This equation is also 11.49 in Marion and Thornton.)

The inertial tensor in the center-of-mass coordinate system is given by

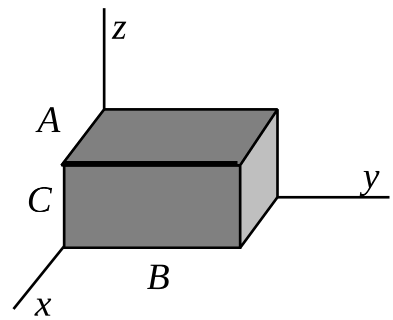
$$I'_{ij} = I_{ij} - T_{ij} \qquad \text{(C.7)}$$

where T_{ij} is the inertia of a point mass M at the center-of-mass's position $\boldsymbol{R}$:

$$T_{ij} = M\left(R^2\delta_{ij} - R_i R_j\right) . \qquad \text{(C.7a)}$$

A.1. Brick

Center of mass: $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} A \\ \frac{1}{2} B \\ \frac{1}{2} C \end{pmatrix}$



The translation tensor is

$$T_{ij} = M\left(R^2\delta_{ij} - R_i R_j\right) = \frac{M}{4}\begin{pmatrix} B^2+C^2 & -AB & -AC \\ -AB & A^2+C^2 & -BC \\ -AC & -AC & A^2+B^2 \end{pmatrix}.$$

The tensors for the brick, calculated in the Worksheet *Inertial Integrals* , are

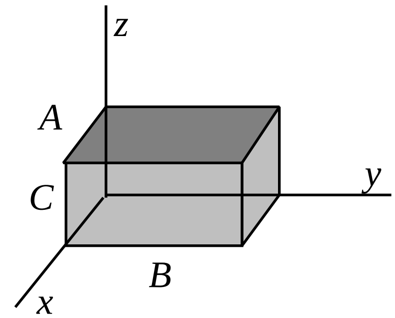
origin at corner, I_{ij} = center of mass, I'_{ij} =

$$\frac{M}{12}\begin{pmatrix} 4(B^2+C^2) & -3AB & -3AC \\ -3AB & 4(A^2+C^2) & -3BC \\ -3AC & -3BC & 4(A^2+B^2) \end{pmatrix}, \quad \frac{M}{12}\begin{pmatrix} B^2+C^2 & 0 & 0 \\ 0 & A^2+C^2 & 0 \\ 0 & 0 & A^2+B^2 \end{pmatrix}.$$

These 3 matrices actually work in eq. (C.7)! √

A.2. Cage

Center of mass: $$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}A \\ \frac{1}{2}B \\ \frac{1}{2}C \end{pmatrix}$$



The translation tensor is the same as for the brick. The tensors calculated in the Worksheet on *Inertial Integrals* are

origin at corner, I_{ij} =

$$\frac{\sigma}{12}\begin{pmatrix} 12AB^2C+8B^3(A+C) + 12ABC^2+8C^3(A+B) & -6AB(AB+BC+AC) & -6AC(AB+BC+AC) \\ -6AB(AB+BC+AC) & 12A^2BC+8A^3(B+C) + 12ABC^2+8C^3(A+B) & -6BC(AB+BC+AC) \\ -6AC(AB+BC+AC) & -6BC(AB+BC+AC) & 12A^2BC+8A^3(B+C) + 12AB^2C+8B^3(A+C) \end{pmatrix}$$

center of mass, I'_{ij} =

$$\frac{\sigma}{6}\begin{pmatrix} B^3(A+C)+3AB^2C + C^3(A+B)+3ABC^2 & 0 & 0 \\ 0 & A^3(B+C)+3A^2BC + C^3(A+B)+3ABC^2 & 0 \\ 0 & 0 & A^3(B+C)+3A^2BC + B^3(A+C)+3AB^2C \end{pmatrix}$$

The difference is

$$I_{ij} - I'_{ij} = \frac{M}{4}\begin{pmatrix} B^2+C^2 & -AB & -AC \\ -AB & A^2+C^2 & -BC \\ -AC & -AC & A^2+B^2 \end{pmatrix}.$$

check comparison:

the difference is equal to T_{ij}

B. TRANSLATING THE CAGEX INERTIAL TENSOR

Use formula (C.7) in the chapter on Rigid Bodies to compare the tensors you calculated in the worksheet *Inertial Integrals*, in the center of mass and another coordinate system.

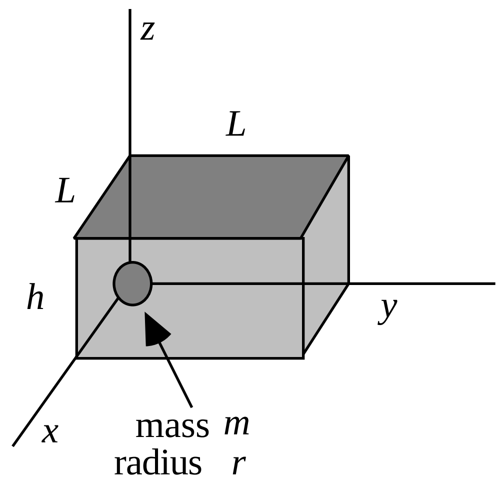
The inertial tensor in the center-of-mass coordinate system is given by

$$I'_{ij} = I_{ij} - T_{ij} \qquad \text{(C.7)}$$

where T_{ij} is the inertia of a point mass M at the center-of-mass's position $\boldsymbol{R}$:

$$T_{ij} = M\left(R^2\delta_{ij} - R_i R_j\right) . \qquad \text{(C.7a)}$$

B.1. From corner of cage to center of mass



The CageX apparatus consists of a hollow rectangular cage with two square faces and four narrow rectangular faces.

Its length in the x and y directions is L, and the length in the z direction is h .

The walls of the cage have a uniform surface mass density, and its total mass is M .

In addition there is a clay sphere of mass m fastened to the corner of the cage at the origin. The radius of this sphere is r .

The expression for the center of mass of the cage-ball system in terms of M , m , L , h , and r is:

$$\boldsymbol{R} = \frac{M}{M+m}\left(\frac{L}{2}, \frac{L}{2}, \frac{h}{2}\right)$$

The numerical values from the *Inertial Integrals* Worksheet are $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \end{pmatrix}$.

B.1. From corner of cage to center of mass (continued)

The translation tensor for the cage-ball apparatus expressed in terms of M , m , L , and h is

$$M_T\left(R^2\delta_{ij} - R_i R_j\right) = T_{ij} = \frac{M^2}{M+m}\,\frac{1}{4}\begin{pmatrix} L^2+h^2 & -L^2 & -Lh \\ -L^2 & L^2+h^2 & -Lh \\ -Lh & -Lh & 2L^2 \end{pmatrix}$$

The inertial tensor for the cage-ball apparatus in the original coordinate system from section **C** of the Worksheet on *Inertial Integrals* expressed in terms of M , m , L , and h is

$$I_{ij} = \frac{M}{24(L^2+2Lh)}\begin{pmatrix} 8L^4+20L^3h \\ +12L^2h^2+16Lh^3 & -6L^3(L+2h) & -6L^2h(L+2h) \\ -6L^3(L+2h) & 8L^4+20L^3h \\ & +12L^2h^2+16Lh^3 & -6L^2h(L+2h) \\ -6L^2h(L+2h) & -6L^2h(L+2h) & 16L^4+40L^3h \end{pmatrix}$$

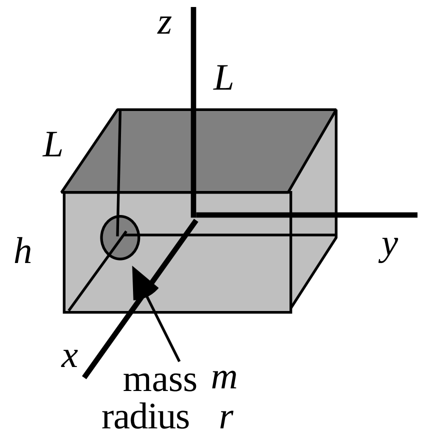
The center-of-mass inertial tensor calculated from these results is I'_{ij} =

$$\frac{Mm}{M+m}\,\frac{1}{4}\begin{pmatrix} L^2+h^2 & -L^2 & -Lh \\ -L^2 & L^2+h^2 & -Lh \\ -Lh & -Lh & 2L^2 \end{pmatrix} +$$

$$\frac{M}{12(L^2+2Lh)}\begin{pmatrix} L^4+L^3h \\ +8Lh^3 & 0 & 0 \\ 0 & L^4+L^3h \\ & +8Lh^3 & 0 \\ 0 & 0 & 2L^4+8L^3h \end{pmatrix}$$

B.2. From center of cage to center of mass

This time we work in a coordinate system with origin at the center of the cage.



The expression for the center of mass of the cage-ball system in terms of M, m, L, and h is:

$$\boldsymbol{R} = -\frac{m}{M+m}\left(\frac{L}{2}, \frac{L}{2}, \frac{h}{2}\right).$$

The numerical values from the *Inertial Integrals* Worksheet are

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{pmatrix}.$$

The translation tensor for the cage-ball apparatus expressed in terms of M, m, L, and h is

$$M_T\left(R^2\delta_{ij} - R_i R_j\right) =$$

$$T_{ij} = \frac{m^2}{M+m}\frac{1}{4}\begin{pmatrix} L^2+h^2 & -L^2 & -Lh \\ -L^2 & L^2+h^2 & -Lh \\ -Lh & -Lh & 2L^2 \end{pmatrix}$$

B.2. From center of cage to center of mass (continued)

The inertial tensor for the **ball only** in the cage-centered coordinate system of section **B** of the Worksheet on *Inertial Integrals* may be approximately expressed in terms of m, L, and h as

$$I_{ij} = \frac{m}{4}\begin{pmatrix} L^2+h^2 & -L^2 & -Lh \\ -L^2 & L^2+h^2 & -Lh \\ -Lh & -Lh & 2L^2 \end{pmatrix}$$

The inertial tensor ,for the **cage only** in the cage-centered coordinate system found in section **B** of the Worksheet on *Inertial Integrals*, expressed in terms of M, m, L, h, and r is

$$I_{ij} = \frac{M}{12(L^2+2Lh)}\begin{pmatrix} L^4+4L^3h+3L^2h^2+2Lh^3 & 0 & 0 \\ 0 & L^4+4L^3h+3L^2h^2+2Lh^3 & 0 \\ 0 & 0 & 2L^4+8L^3h \end{pmatrix}$$

The inertial tensor for the **cage plus ball** in the cage-centered coordinate system of section **B** of the Worksheet on *Inertial Integrals*, expressed in terms of m, L, and h is therefor

$$I_{tot} = = I_{ball} + I_{cage} = \frac{m}{4}\begin{pmatrix} L^2+h^2 & -L^2 & -Lh \\ -L^2 & L^2+h^2 & -Lh \\ -Lh & -Lh & 2L^2 \end{pmatrix} +$$

$$\frac{M}{12(L^2+2Lh)}\begin{pmatrix} L^4+4L^3h+3L^2h^2+2Lh^3 & 0 & 0 \\ 0 & L^4+4L^3h+3L^2h^2+2Lh^3 & 0 \\ 0 & 0 & 2L^4+8L^3h \end{pmatrix}$$

B.2. From center of cage to center of mass (concluded)

The translation tensor for the cage-ball apparatus, expressed in terms of M , m , L , and h , found on page 6, is

$$T_{ij} = \frac{m^2}{M+m}\,\frac{1}{4}\begin{pmatrix} L^2+h^2 & -L^2 & -Lh \\ -L^2 & L^2+h^2 & -Lh \\ -Lh & -Lh & 2L^2 \end{pmatrix}$$

The inertial tensor for the **cage plus ball** in the cage-centered coordinate system, calculated on the previous page, expressed in terms of m , L , and h is

$$I_t = \frac{m}{4}\begin{pmatrix} L^2+h^2 & -L^2 & -Lh \\ -L^2 & L^2+h^2 & -Lh \\ -Lh & -Lh & 2L^2 \end{pmatrix} +$$

$$\frac{M}{12(L^2+2Lh)}\begin{pmatrix} L^4+4L^3h+3L^2h^2+2Lh^3 & 0 & 0 \\ 0 & L^4+4L^3h+3L^2h^2+2Lh^3 & 0 \\ 0 & 0 & 2L^4+8L^3h \end{pmatrix}$$

The center-of-mass inertial tensor calculated from these results is I'_{ij} =

$$\frac{Mm}{M+m}\,\frac{1}{4}\begin{pmatrix} L^2+h^2 & -L^2 & -Lh \\ -L^2 & L^2+h^2 & -Lh \\ -Lh & -Lh & 2L^2 \end{pmatrix} +$$

$$\frac{M}{12(L^2+2Lh)}\begin{pmatrix} L^4+L^3h+8Lh^3 & 0 & 0 \\ 0 & L^4+L^3h+8Lh^3 & 0 \\ 0 & 0 & 2L^4+8L^3h \end{pmatrix}$$

Comparing this result to that of part **B.1**, we observe that they are the same.

C. Numerical values

The inertial tensor for the cage-ball apparatus in the original coordinate system, found in section **C.3** of Worksheet 3 *Inertial Integrals*, is I_{ij} =

numerically expressed in units of____ .

The measured numerical values of the center-of-mass coordinates, from page 5 of the CageX1 Workbook, are

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{pmatrix}.$$

The corresponding translation tensor for the cage-ball apparatus, evaluated numerically from the expression

$T_{ij} = M\left(R^2\delta_{ij} - R_i R_j\right)$ is T_{ij} =

The center-of-mass inertial tensor calculated from these results, is I'_{ij} =

numerically expressed in units of ____.

B.3. Numerical values continued

We can compare some of the elements of the inertial tensor to the measured results from the CageX1 Workbook, page 10:

Cases	Observed moment I'_{obs} (units)	Corresponding element $I'_{??}$	Calculated value above (units)	Comparison

CageX Inertia Tensor in Center-of-Mass frame (correction to Translating Tensors worksheet solution, p. 5 and p. 8, to Rotating Tensors worksheet solution, p. 4, and to Principal Axes worksheet solution, p. 6)

Written as one piece:

$$\begin{pmatrix} \frac{M\,(3\,h^2\,L\,(2\,m+M)+2\,h^3\,(4\,m+M)+L^3\,(4\,m+M)+2\,h\,L^2\,(5\,m+2\,M))}{12\,(2\,h+L)\,(m+M)} & -\frac{L^2\,m\,M}{4\,(m+M)} & -\frac{h\,L\,m\,M}{4\,(m+M)} \\ -\frac{L^2\,m\,M}{4\,(m+M)} & \frac{M\,(3\,h^2\,L\,(2\,m+M)+2\,h^3\,(4\,m+M)+L^3\,(4\,m+M)+2\,h\,L^2\,(5\,m+2\,M))}{12\,(2\,h+L)\,(m+M)} & -\frac{h\,L\,m\,M}{4\,(m+M)} \\ -\frac{h\,L\,m\,M}{4\,(m+M)} & -\frac{h\,L\,m\,M}{4\,(m+M)} & \frac{L^2\,M\,(10\,h\,m+4\,L\,m+4\,h\,M+L\,M)}{6\,(2\,h+L)\,(m+M)} \end{pmatrix}$$

Written as two pieces:

$$\frac{m\,M}{4\,(m+M)} \begin{pmatrix} h^2+L^2 & -L^2 & -h\,L \\ -L^2 & h^2+L^2 & -h\,L \\ -h\,L & -h\,L & 2\,L^2 \end{pmatrix} +$$

$$\frac{M}{12\,(2\,h\,L+L^2)} \begin{pmatrix} 2\,h^3\,L+3\,h^2\,L^2+4\,h\,L^3+L^4 & 0 & 0 \\ 0 & 2\,h^3\,L+3\,h^2\,L^2+4\,h\,L^3+L^4 & 0 \\ 0 & 0 & 8\,h\,L^3+2\,L^4 \end{pmatrix}$$