solutions to
ROTATING VECTORS
worksheet with worked example

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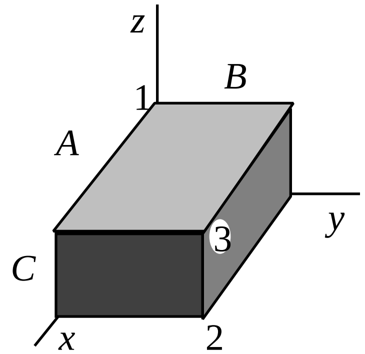
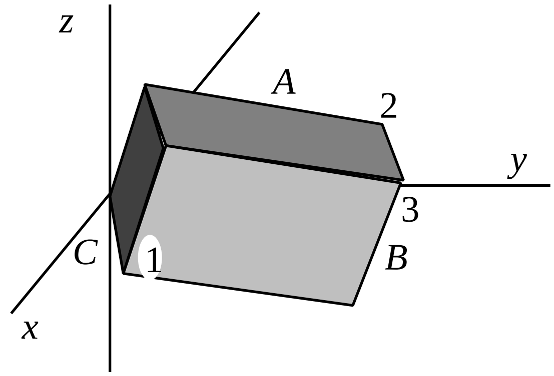
TOPICS page

A. WORKED EXAMPLE: THE RECTANGULAR BRICK



A rectangular box lies in the first octant with its corner at the origin and its edges parallel to the coordinate axes. Its dimensions in the *x, y,* and *z* directions are *A, B,* and *C* respectively. *A* is the longest side and *C* is the shortest side.



Keeping the corner at the origin fixed, the box is to be rotated so that it has its longest diagonal along the +*y* axis, and two of its shortest edges in the *xy* plane. The short edge adjacent to the origin is in the first quadrant of the *xy* plane.

A.1. Equations for the direction cosines

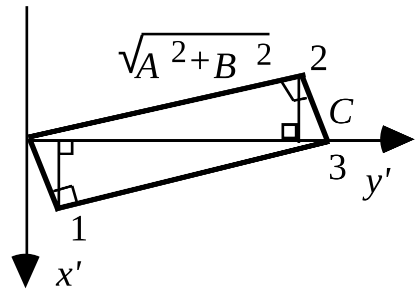
A rotation is given in terms of its direction cosines R_{ij} :

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}.$$

Choose two corners which, together with the corner at the origin, can determine the orientation of the box. Write their coordinates before and after the rotation. Which corners will be easiest to use?

The corners associated with the edges and diagonal whose final positions are specified will be the easiest to use.

To find their coordinates after rotation, sketch a cross section in the $x'y'$ plane for $z' = 0$. The cross section is a rectangle whose short side is C, and whose longest side is the diagonal of a face with sides A and B. Dropping a perpendicular from a corner to the x' axis, as shown, gives a similar right triangle whose legs are the x' and y' coordinates of the two points to be determined:



For point 1: $\dfrac{x'_1}{C} = \dfrac{\sqrt{A^2+B^2}}{\sqrt{A^2+B^2+C^2}}$, $\dfrac{y'_1}{C} = \dfrac{C}{\sqrt{A^2+B^2+C^2}}$

For point 2: $\dfrac{-x'_2}{\sqrt{A^2+B^2}} = \dfrac{C}{\sqrt{A^2+B^2+C^2}}$, $\dfrac{y'_2}{\sqrt{A^2+B^2}} = \dfrac{\sqrt{A^2+B^2}}{\sqrt{A^2+B^2+C^2}}$

Checking the results: $x'_2 = -x'_1$, and $y'_1+y'_2 = \sqrt{A^2+B^2+C^2}$, as expected

Corner	Coordinates before rotation	Coordinates after rotation
1. top, back left	$(0, 0, C)$	$\left(\dfrac{C\sqrt{A^2+B^2}}{\sqrt{A^2+B^2+C^2}}, \dfrac{C^2}{\sqrt{A^2+B^2+C^2}}, 0\right)$
2. bottom, front right	$(A, B, 0)$	$\left(\dfrac{-C\sqrt{A^2+B^2}}{\sqrt{A^2+B^2+C^2}}, \dfrac{A^2+B^2}{\sqrt{A^2+B^2+C^2}}, 0\right)$

For each vector, we can write equations for the new coordinates in terms of the old ones, using direction cosines R_{ij}. Then we can set these expressions equal to the desired positions of the corners after rotations.

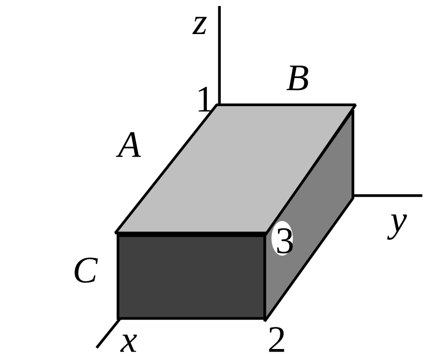
For point 1: $R_{13}\, C = \dfrac{C\sqrt{A^2+B^2}}{\sqrt{A^2+B^2+C^2}}$, $R_{23}C = \dfrac{C^2}{\sqrt{A^2+B^2+C^2}}$, $R_{33}C = 0$

For point 2:

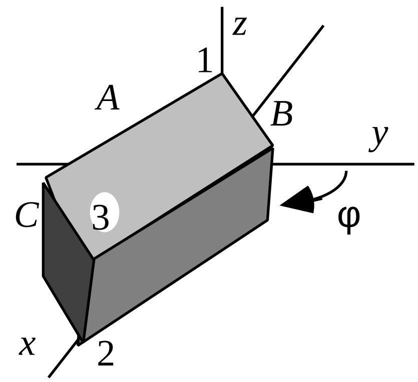
$$R_{11}A + R_{12}B = \frac{-C\sqrt{A^2+B^2}}{\sqrt{A^2+B^2+C^2}}, \quad R_{21}A+R_{22}B = \frac{A^2+B^2}{\sqrt{A^2+B^2+C^2}}, \quad R_{31}A+R_{32}B = 0$$

These are six linear equations which the nine coefficients R_{ij} muse satisfy. To get three more we would have to make a similar construction in another plane. Then we could solve the linear equations to find the cosines. Instead we will find the transformation from geometric reasoning, deconstructing it into a sequence of simpler rotations, each of which keeps one of the coordinate axes fixed.

A.2. Finding the Euler angles



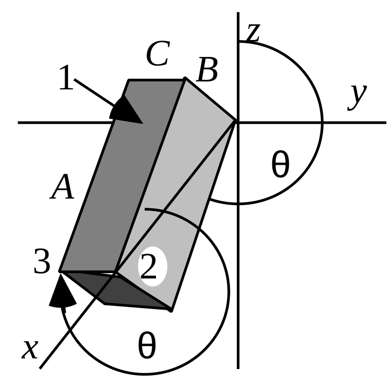
A more intuitive way to find the rotation matrix is to figure out how to make the rotation as a sequence of two-dimensional rotations which leave one axis fixed. The Euler angles conventionally rotate about the *z*, *x* , and *z* axes in that order.



First a rotation about the *z* axis, to bring the long diagonal O3 into the *xz* plane.The angle of this rotation, φ, is the angle originally between the side diagonal O2 and the *y* axis:

$$\sin\varphi = B/\sqrt{A^2+B^2}\ ,\ \ \cos\varphi = A/\sqrt{A^2+B^2}\ .$$

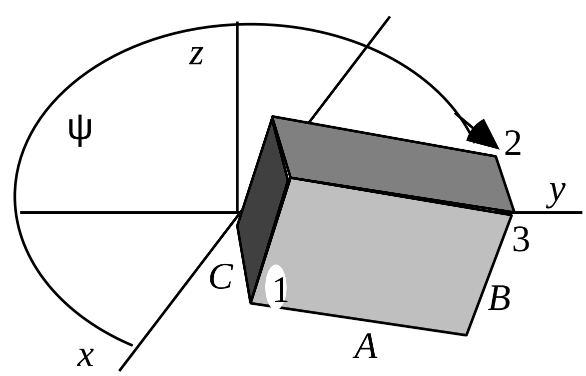
The angle is positive: the axes rotate counterclockwise, the body goes clockwise.



Next a rotation about the *x* axis, to bring edge O1, originally along the *z* axis, into the *xy* plane, actually along the –*y* axis. The angle of this rotation, θ, is the angle between the *z* axis and the *x* axis,

$$\sin\theta = -1,\ \ \cos\theta = 0.$$

The sign comes by considering what happpens to points 1 and 3.



Finally a rotation about the *z* axis, to bring the long diagonal O3 (now in the *xy* plane) to the *y* axis. The angle of this rotation, ψ , is π/2 greater than the angle between the long diagonal O3 and the face diagonal O2, so

s axis.B^2+C^2)

$$\cos\psi = -C/\sqrt{A^2+B^2+C^2}\ ,$$

Once again the signs are obtained by considering what happens to points 1 and 2.

Altogether the rotation matrix is given by

$$R = R_\psi R_\theta R_\varphi = \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix}$$

$$= \frac{1}{\sqrt{A^2+B^2}\sqrt{A^2+B^2+C^2}} \begin{pmatrix} -CA & -BC & (A^2+B^2) \\ A\sqrt{A^2+B^2} & B\sqrt{A^2+B^2} & C\sqrt{A^2+B^2} \\ -B\sqrt{A^2+B^2+C^2} & A\sqrt{A^2+B^2+C^2} & 0 \end{pmatrix}$$

A. 3. Checking the direction cosines

We can show that our result for R_{ij} satisfies the equations we found in part A.1:

For point 1: $R_{13}\,C = \dfrac{C\sqrt{A^2+B^2}}{\sqrt{A^2+B^2+C^2}}$, $R_{23}C = \dfrac{C^2}{\sqrt{A^2+B^2+C^2}}$, $R_{33}C = 0$

For point 2:

$$R_{11}A + R_{12}B = \frac{-C\sqrt{A^2+B^2}}{\sqrt{A^2+B^2+C^2}}, \quad R_{21}A + R_{22}B = \frac{A^2+B^2}{\sqrt{A^2+B^2+C^2}}, \quad R_{31}A + R_{32}B = 0$$

Substituting the values of the R_{ij} found above can check these equations, or you can think of it as a check on the values of R_{ij}.

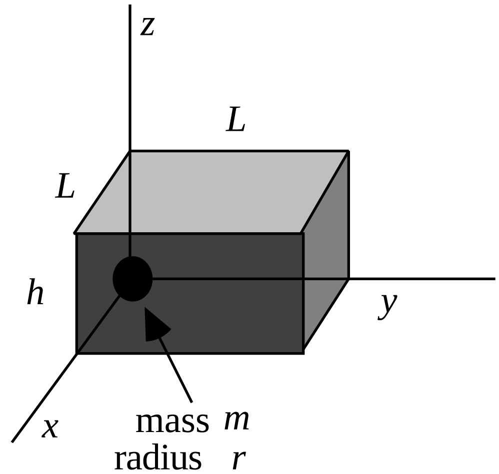
B. APPLICATION TO CAGELAB APPARATUS - ALGEBRA

The CageLab apparatus consists of a hollow rectangular cage with two square faces and four narrow rectangular faces.

Its length in the x and y directions is L, and the length in the z direction is h .

The walls of the cage have a uniform surface mass density, and its total mass is M .

In addition there is a clay sphere of mass m fastened to the corner of the cage at the origin. The radius of this sphere is r .



For this exercise we want to find a rotation which keeps the sphere fixed and rotates the cage so that its long diagonal is along the z axis.

B. 1. Equation for the direction cosines

before the rotation, where is the diagonal point P located?

$$(x_P, y_P, z_P) = (L , L , h) .$$

After the rotation, where will the diagonal point P be located?

$$(x'_P, y'_P, z'_P) = (0 , 0 , \sqrt{2L^2+h^2}) .$$

Use this result to find three linear equations for the nine direction cosines R_{ij} :

$$R_{11}L + R_{12}L + R_{13}h = 0$$

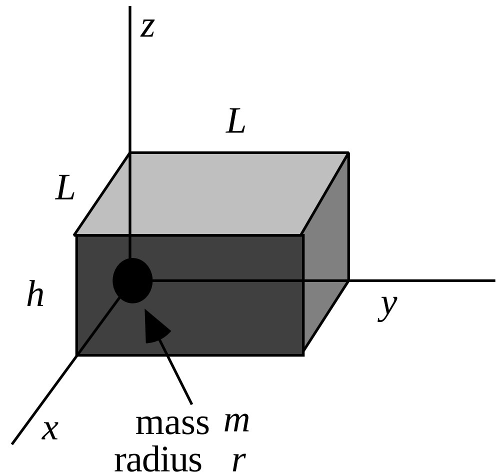
$$R_{21}L + R_{22}L + R_{23}h = 0$$

$$R_{31}L + R_{32}L + R_{33}h = \sqrt{2L^2+h^2}$$

B. 2. Euler angles for initial rotations

We want to find a rotation which keeps the sphere fixed and rotates the cage so that its long diagonal is along the z axis.

We can do this in only two steps, leaving the final rotation angle ψ to be determined later.



The first rotation is about the z axis.

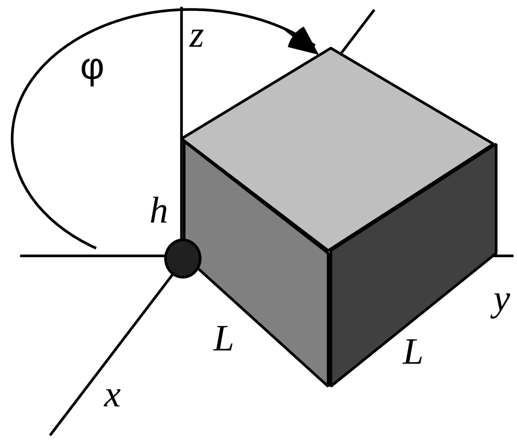
The angle φ is chosen so that

the long diagonal is in the yz plane, and the side diagonal is along the $+y$ axis

The resulting angle φ satisfies

$$\sin\varphi = \frac{L}{\sqrt{L^2+L^2}} = -1/\sqrt{2}$$

$$\cos\varphi = 1/\sqrt{2}$$



The second rotation is about the x axis.

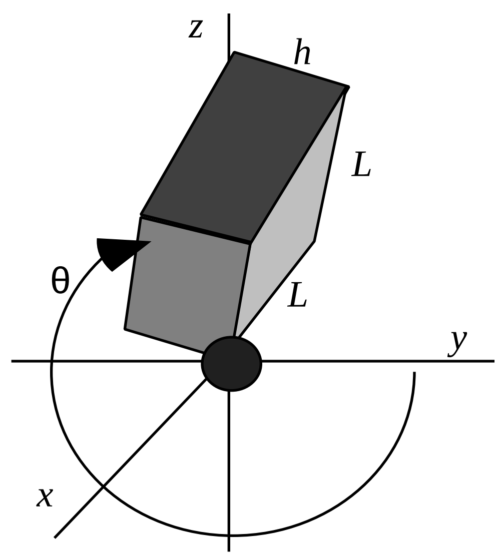
The angle θ is chosen so that

the long diagonal is brought to the z axis

The resulting angle θ satisfies

$$\sin\theta = \frac{-\sqrt{2}L}{\sqrt{2L^2+h^2}}$$

$$\cos\theta = \frac{h}{\sqrt{2L^2+h^2}}$$



These values can be substituted in the general expression for the rotation matrix R_{ij},

$$R = R_\psi R_\theta R_\varphi = \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix}.$$

The result will still contain the last angle ψ as an undetermined parameter:

$$R = \begin{pmatrix} \frac{\cos\psi}{\sqrt{2}} + \frac{h\sin\psi}{\sqrt{2(2L^2+h^2)}} & \frac{-\cos\psi}{\sqrt{2}} + \frac{h\sin\psi}{\sqrt{2(2L^2+h^2)}} & \frac{-\sqrt{2}L\sin\psi}{\sqrt{2L^2+h}} \\ \frac{-\sin\psi}{\sqrt{2}} + \frac{h\cos\psi}{\sqrt{2(2L^2+h^2)}} & \frac{\sin\psi}{\sqrt{2}} + \frac{h\cos\psi}{\sqrt{2(2L^2+h^2)}} & \frac{-\sqrt{2}L\cos\psi}{\sqrt{2L^2+h}} \\ \frac{L}{\sqrt{2L^2+h^2}} & \frac{L}{\sqrt{2L^2+h^2}} & \frac{h}{\sqrt{2L^2+h}} \end{pmatrix}$$

B. 3. Checking the direction cosines

Check that your result satisfies the equations you derived in part **B.1**, no matter what ψ is.

$R_{11}L + R_{12}L + R_{13}h =$

$$= \left(\frac{\cos\psi}{\sqrt{2}} + \frac{h\sin\psi}{\sqrt{2(2L^2+h^2)}}\right)L + \left(\frac{-\cos\psi}{\sqrt{2}} + \frac{h\sin\psi}{\sqrt{2(2L^2+h^2)}}\right)L + \left(\frac{-\sqrt{2}L\sin\psi}{\sqrt{2L^2+h}}\right)h = 0 \;\surd$$

$R_{21}L + R_{22}L + R_{23}h =$

$$= \left(\frac{-\sin\psi}{\sqrt{2}} + \frac{h\cos\psi}{\sqrt{2(2L^2+h^2)}}\right)L + \left(\frac{\sin\psi}{\sqrt{2}} + \frac{h\cos\psi}{\sqrt{2(2L^2+h^2)}}\right)L + \left(\frac{-\sqrt{2}L\cos\psi}{\sqrt{2L^2+h}}\right)h = 0 \;\surd$$

$R_{31}L + R_{32}L + R_{33}h =$

$$= \left(\frac{L}{\sqrt{2L^2+h^2}}\right)L + \left(\frac{L}{\sqrt{2L^2+h^2}}\right)L + \left(\frac{h}{\sqrt{2L^2+h}}\right)h = \sqrt{2L^2+h^2} \;\surd$$