solutions to
ROTATING TENSORS
worksheet

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TOPICS page

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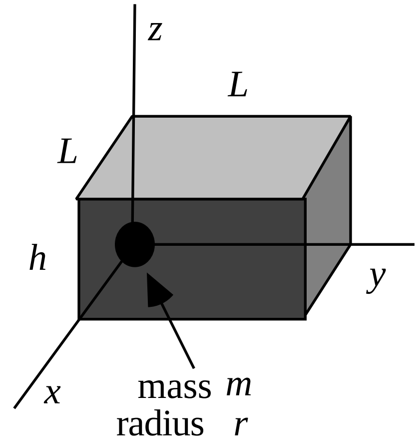
A. APPLYING THE PREVIOUS ROTATION – ALGEBRA



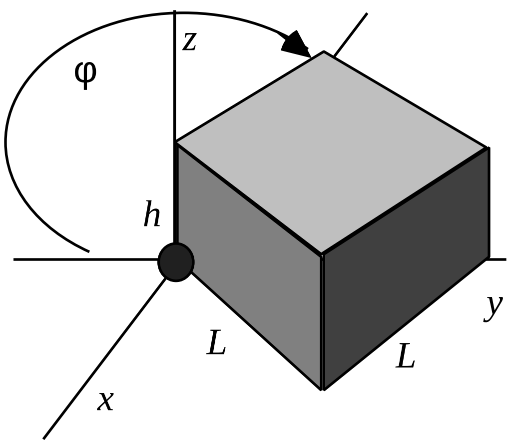
The CageLab apparatus consists of a hollow rectangular cage with two square faces.

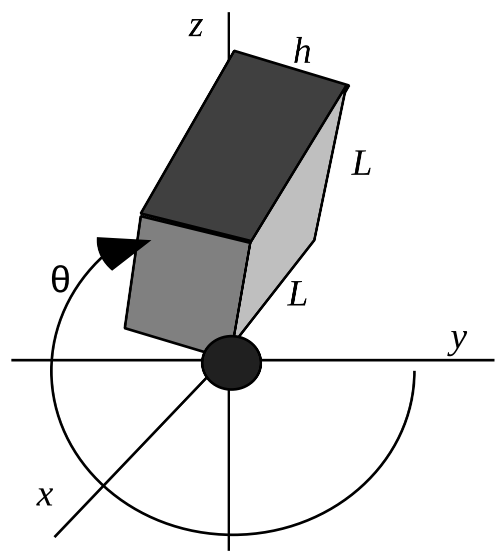
Its length in the x and y directions is L, and the length in the z direction is h . h is shorter than L.

The walls of the cage have a uniform surface mass density, and its total mass is M .

There is a clay sphere of mass m and radius r fastened to the corner of the cage at the origin.

In the worksheet *Rotating Vectors* we found a rotation which keeps the sphere fixed and rotates the cage so that its long diagonal is along the z axis.





First, there was a rotation about the z axis by an angle φ, such that

$$\sin\varphi = -\sqrt{2}/2 \qquad \cos\varphi = \sqrt{2}/2$$

Next, there was a rotation about the x axis by an angle θ such that

$$\sin\theta = \frac{-\sqrt{2}L}{\sqrt{2L^2+h^2}} \qquad \cos\theta = \frac{h}{\sqrt{2L^2+h^2}}$$

These values can be substituted in the general expression for the rotation matrix ,

$$R = R_\psi R_\theta R_\varphi = \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix}$$

The result still contains the last angle ψ as an undetermined parameter: in terms of *L, h, etc.*, we found

$R(\psi) =$

$$\begin{pmatrix} \frac{\cos\psi}{\sqrt{2}} + \frac{h\sin\psi}{\sqrt{2(2L^2+h^2)}} & \frac{-\cos\psi}{\sqrt{2}} + \frac{h\sin\psi}{\sqrt{2(2L^2+h^2)}} & \frac{-\sqrt{2}L\sin\psi}{\sqrt{2L^2+h^2}} \\ \frac{-\sin\psi}{\sqrt{2}} + \frac{h\cos\psi}{\sqrt{2(2L^2+h^2)}} & \frac{\sin\psi}{\sqrt{2}} + \frac{h\cos\psi}{\sqrt{2(2L^2+h^2)}} & \frac{-\sqrt{2}L\cos\psi}{\sqrt{2L^2+h^2}} \\ \frac{L}{\sqrt{2L^2+h^2}} & \frac{L}{\sqrt{2L^2+h^2}} & \frac{h}{\sqrt{2L^2+h^2}} \end{pmatrix}$$

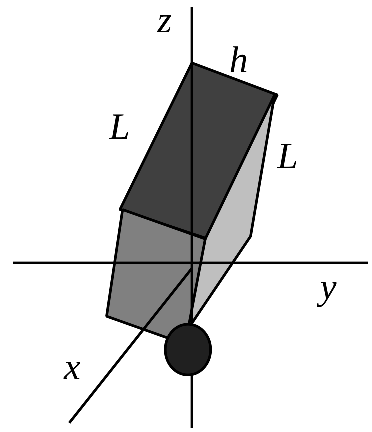
For simplicity, we can choose the case ψ = 0. Then, in terms of *L*, *h*, *etc.*,

$$R(\psi=0) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{h}{\sqrt{2(2L^2+h^2)}} & \frac{h}{\sqrt{2(2L^2+h^2)}} & \frac{-\sqrt{2}L}{\sqrt{2L^2+h^2}} \\ \frac{L}{\sqrt{2L^2+h^2}} & \frac{L}{\sqrt{2L^2+h^2}} & \frac{h}{\sqrt{2L^2+h^2}} \end{pmatrix}$$

We can apply the same rotation to the apparatus in its center of mass.

The result will also have the ball on the z axis, but the center of mass will be at the origin.

The inertial tensor for the combined ball plus cage, **in their joint center of mass**, was found in the Worksheet *Translating Tensors* :



$$I_{tot} = I_{ball} + I_{cage} = \quad \text{expression in terms of } L,\ h, \text{ etc.}$$

$$= \frac{m}{4}\begin{pmatrix} L^2+h^2 & -L^2 & -Lh \\ -L^2 & L^2+h^2 & -Lh \\ -Lh & -Lh & 2L^2 \end{pmatrix} +$$

$$\frac{M}{12(L^2+2Lh)}\begin{pmatrix} L^4+4L^3h+3L^2h^2+2Lh^3 & 0 & 0 \\ 0 & L^4+4L^3h+3L^2h^2+2Lh^3 & 0 \\ 0 & 0 & 2L^4+8L^3h \end{pmatrix}$$

The rotated inertial tensor $R\ I'\ R^{\mathrm{T}}$ is calculated in two steps.

First, multiply I' times R^{T} to compute $I'R^{\mathrm{T}}$: (expressed in terms of L, h, *etc.*)

$$\frac{M}{12(L^2+2Lh)}\ \frac{1}{\sqrt{2(2L^2+h^2)}}\begin{pmatrix} \sqrt{2L^2+h^2}(L^4+4L^3h+3L^2h^2+2Lh^3) & h(L^4+4L^3h+3L^2h^2+2Lh^3) & \sqrt{2}L(L^4+4L^3h+3L^2h^2+2Lh^3) \\ -\sqrt{2L^2+h^2}(L^4+4L^3h+3L^2h^2+2Lh^3) & h(L^4+4L^3h+3L^2h^2+2Lh^3) & \sqrt{2}L(L^4+4L^3h+3L^2h^2+2Lh^3) \\ 0 & -4L^4(L+4h) & 2\sqrt{2}L^3h(L+4h) \end{pmatrix}$$

$$+\ \frac{m}{4\sqrt{2(2L^2+h^2)}}\begin{pmatrix} (2L^2+h^2)\sqrt{2L^2+h^2} & 2L^2h+h^3 & 0 \\ -(2L^2+h^2)\sqrt{2L^2+h^2} & 2L^2h+h^3 & 0 \\ 0 & -2L(2L^2+h^2) & 0 \end{pmatrix}$$

Next, multiply on the left by R to get $R\ I'\ R^{\mathrm{T}}$: (expressed in terms of L, h, *etc.*)

$$\frac{M}{12(L^2+2Lh)}\ \frac{1}{2L^2+h^2}\begin{pmatrix} (2L^2+h^2)(L^4+4L^3h+3L^2h^2+2Lh^3) & 0 & 0 \\ 0 & 4L^6+16L^5h+L^4h^2+4L^3h^3+3L^2h^4+2Lh^5 & \sqrt{2}(-L^5h-4L^4h^2+3L^3h^3+2L^2h^4) \\ 0 & \sqrt{2}(-L^5h-4L^4h^2+3L^3h^3+2L^2h^4) & 2L^6+8L^5h+4L^4h^2-4L^3h^3 \end{pmatrix}$$

$$+\ \frac{m}{4}\ \frac{1}{2(2L^2+h^2)}\begin{pmatrix} (2L^2+h^2)^2 & 0 & 0 \\ 0 & 2(2L^2+h^2)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Remarks on this result: <u>the x axis is a principal axis: *Ixy* = 0 = *Ixz*</u>