PRINCIPAL AXES

Worksheet 6

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TOPICS page

Some students may wish to work out the numbers before completing the algebra.

A. EXPLOITING SYMMETRY TO FIND A PRINCIPAL AXIS

A.1. Reflection symmetry

There are many mathematical methods for finding the rotation that diagonalizes a tensor.

For example, as discussed in the Mathematical Appendix of the Notes, the rotation matrix has a simple expression in terms of the eigenvectors (eq. **B.6**). Computer algebra program packages, such as Maple, Mathematica, MatLab, or (for professional UNIX/LINUX based systems) the CERN mathematical library, contain programs to find the eigenvalues and eigenvectors of a matrix. The eigenvectors are the rows of the rotation matrix, and the eigenvalues are the corresponding entries in the diagonalized tensor expressed in the rotated basis.

In some cases we can make the task more understandable by helping out with some physical insight. What's needed is **symmetry**!

The main example is when an object possesses a plane of **reflection symmetry**. The plane of symmetry necessarily passes through the center of mass.

In a coordinate system with **the origin in the plane of symmetry**, an axis **perpendicular to the plane of symmetry** is a principal axis!

For example, suppose the plane $z = 0$ is a plane of symmetry. Then

$$\rho(x, y, z) = \rho(x, y, -z) .$$

This can be used to show that the off-diagonal matrix elements I_{xz} and I_{yz} are 0.

When one principal axis is found, the other two are not hard to find. They line in the plane perpendicular to the known principal axis. The Euler angles of the rotation which diagonalizes the intertial tensor follow. The first two Euler angles φ and θ can be used to bring the known axis to the z direction; then the last angle ψ diagonalizes the remaining 2×2 submatrix in the x-y plane:

1. Rotate by φ around z to bring the known axis into the y-z plane.

2. Rotate by θ around x to bring the known axis into the $+z$ direction.

3. Rotate by ψ around z to diagonalize the submatrix $\begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix}$. This is much easier than the full 3×3 matrix.

A.2. Application to CageX apparatus

a. The CageX apparatus of cage + ball has a plane of symmetry. Thus we can place a principal axis along the z direction after two rotations.

Describe the plane of symmetry

i. in the original coordinate system before the rotations

It is the plane $x = y$

ii. in a coordinate-system independent way, *i.e.* without using **any** coordinate system.

It includes the principal diagonal and diagonals of both square faces, *i.e.*

it includes the ball and the short edges nearest to and farthest from the ball.

b. Describe the corresponding principal axis, in terms of the cage and ball,

i. in the original coordinate system before the rotations

It goes through the center of mass,
in a plane with z = constant, *i.e.* parallel to the plane $z = 0$.
in a direction parallel to the plane $x = -y$, *i.e.* at an angle of $-45^{\text{o}} = -\frac{\pi}{4}$
the equation for the line is

$$x + y = x_{\text{cm}} + y_{\text{cm}}$$

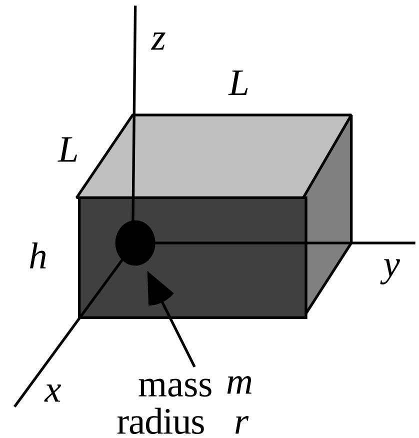
ii. in a coordinate-system independent way, *i.e.* without using **any** coordinate system.

It goes through the center of mass,
parallel to the diagonal of the square face that doesn't include the short side where the ball is.

B. FINDING THE PRINCIPAL AXES FOR CAGEX

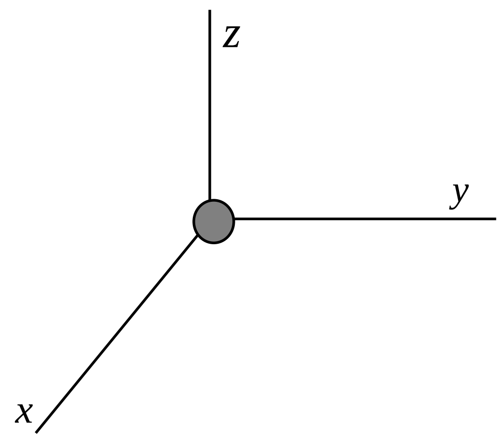
Now that we have identified the symmetry plane and the corresponding principal axis, we are ready to find the other two principal axes.

First, we have to rotate the coordinate system so that the principal axis is in the *z* direction. We do this with the first two Euler angles: We can find them by geometric reasoning.



The geometry is easiest in the original ball-centered coordinate system, before the translation to the center of mass. After we find the needed rotation, we will apply it to the center-of-mass inertial tensor I'

The first rotation is about the *z* axis.



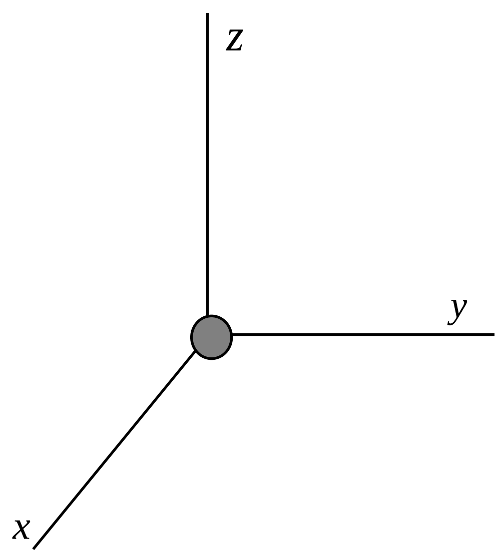
The angle φ is chosen so that the known principal axis is **perpendicular to** the *x* direction.

The resulting angle φ satisfies

$$\sin\varphi = 1/\sqrt{2}$$

$$\cos\varphi = 1/\sqrt{2}$$

The second rotation is about the *x* axis.



The angle θ is chosen so that the known principal axis is parallel to the *z* axis.

The resulting angle θ satisfies

$$\sin\theta = 1$$

$$\cos\theta = 0$$

These values can be substituted in the general expression for the rotation matrix R_{ij},

$$R = R_\psi R_\theta R_\varphi = \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix}.$$

The result will still contain the last angle ψ as an undetermined parameter.

The sines and cosines are:

$\sin\varphi =$ $\sin\theta =$

$\cos\varphi =$ $\cos\theta =$

The rotation matrix is:

$$R = \begin{pmatrix} \cos\psi/\sqrt{2} & \cos\psi/\sqrt{2} & \sin\psi \\ -\sin\psi/\sqrt{2} & -\sin\psi/\sqrt{2} & \cos\psi \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

B.1. Algebra

To determine the last angle ψ, we can use R to rotate the center-of-mass inertial tensor I' and then choose ψ so that the result is diagonal. The diagonal matrix elements will then be the principal moments of inertia.

We found I' in the worksheet Translating Tensors, page _____.

In terms of L, h, m, and M, it is

$$\frac{Mm}{M+m}\,\frac{1}{4}\begin{pmatrix} L^2+h^2 & -L^2 & -Lh \\ -L^2 & L^2+h^2 & -Lh \\ -Lh & -Lh & 2L^2 \end{pmatrix} + \frac{M}{12(L^2+2Lh)}\begin{pmatrix} L^4+L^3h+8Lh^3 & 0 & 0 \\ 0 & L^4+L^3h+8Lh^3 & 0 \\ 0 & 0 & 2L^4+8L^3h \end{pmatrix}$$

We carry out the rotation in two steps. First, we multiply times R to get $R\,I'$:

$R\,I' =$

$$\frac{Mm}{M+m}\,\frac{1}{4}\begin{pmatrix} h^2\cos\psi/\sqrt{2}-Lh\sin\psi & h^2\cos\psi/\sqrt{2}-Lh\sin\psi & -\sqrt{2}Lh\cos\psi+2L^2\sin\psi \\ -h^2\sin\psi/\sqrt{2}-Lh\cos\psi & -h^2\sin\psi/\sqrt{2}-Lh\cos\psi & \sqrt{2}Lh\sin\psi+2L^2\cos\psi \\ (2L^2+h^2)/\sqrt{2} & -(2L^2+h^2)/\sqrt{2} & 0 \end{pmatrix}$$

$$+\ \frac{M}{12(L^2+2Lh)}\begin{pmatrix} \left(L^4+L^3h+8Lh^3\right)\frac{\cos\psi}{\sqrt{2}} & \left(L^4+L^3h+8Lh^3\right)\frac{\cos\psi}{\sqrt{2}} & (2L^4+8L^3h)\sin\psi \\ -\left(L^4+L^3h+8Lh^3\right)\frac{\sin\psi}{\sqrt{2}} & -\left(L^4+L^3h+8Lh^3\right)\frac{\sin\psi}{\sqrt{2}} & (2L^4+8L^3h)\cos\psi \\ \left(L^4+L^3h+8Lh^3\right)\frac{1}{\sqrt{2}} & -\left(L^4+L^3h+8Lh^3\right)\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Next, we multiply by the transpose R^{T} to get the rotated inertial tensor:

$$R^{\mathrm{T}} = \begin{pmatrix} \cos\psi/\sqrt{2} & -\sin\psi/\sqrt{2} & 1/\sqrt{2} \\ \cos\psi/\sqrt{2} & -\sin\psi/\sqrt{2} & -1/\sqrt{2} \\ \sin\psi & \cos\psi & 0 \end{pmatrix}.$$

$R\, I'\, R^{\mathrm{T}}$ =

$$\frac{Mm}{M+m}\frac{1}{4}\begin{pmatrix} 2L^2\sin^2\psi + h^2\cos^2\psi - \sqrt{8}Lh\sin\psi\cos\psi & \sqrt{2}Lh(\sin^2\psi - \cos^2\psi) + (2L^2-h^2)\sin\psi\cos\psi & 0 \\ \sqrt{2}Lh(\sin^2\psi - \cos^2\psi) + (2L^2-h^2)\sin\psi\cos\psi & h^2\sin^2\psi - 2L^2\cos^2\psi + \sqrt{8}Lh\sin\psi\cos\psi & 0 \\ 0 & 0 & 2L^2+h^2 \end{pmatrix}$$

$$+ \frac{M}{12(L^2+2Lh)}\begin{pmatrix} \left(\begin{matrix} L^4+L^3h \\ +8Lh^3 \end{matrix}\right)\cos^2\psi + (2L^4+8L^3h)\sin^2\psi & \left(\begin{matrix} L^4+7L^3h \\ +8Lh^3 \end{matrix}\right)\sin\psi\cos\psi & 0 \\ \left(\begin{matrix} L^4+7L^3h \\ +8Lh^3 \end{matrix}\right)\sin\psi\cos\psi & \left(\begin{matrix} L^4+L^3h \\ +8Lh^3 \end{matrix}\right)\sin^2\psi + (2L^4+8L^3h)\cos^2\psi & 0 \\ 0 & 0 & L^4+L^3h+8Lh^3 \end{pmatrix}$$

The I_{xz} and I_{yz} should both vanish for every value of ψ .

Why should we expect this result?

Because the coordinates have been chosen so that the symmetry plane is $z = 0$

Does your expression for $R\, I'\, R^{\mathrm{T}}$ fulfill this expectation?

yes √ also it is symmetric, as it must be, which helps to check the algebra

The remaining off-diagonal elements $I_{xy} = I_{yx}$ are a function of the undetermined rotation angle ψ. What condition must ψ fulfill in order to make I_{xy} vanish, too?

Let $D \equiv \frac{Mm}{M+m} \frac{Lh}{\sqrt{8}}$ and $E \equiv \frac{Mm}{M+m} \frac{2L^2-h^2}{4} + \frac{M(L^4+7L^3h+8Lh^3)}{12(L^2+2Lh)}$, both > 0 ,

then $D(\sin^2\psi - \cos^2\psi) + E \sin\psi \cos\psi = 0$ $\quad\quad$ $\tan^2\psi + (E/D)\tan\psi - 1 = 0$.

Use this result to evaluate the direction cosines: $\tan\psi = -\frac{E}{2D} \pm \sqrt{\left(\frac{E}{2D}\right)^2 + 1}$

two roots:

+ sign gives positive ψ

– sign gives larger negative ψ

choose positive root

$$\cos\psi = \frac{1}{\sqrt{1+\tan^2\psi}}$$

other root corresponds to $\psi - \frac{\pi}{2}$

$$\sin\psi = \frac{\tan\psi}{\sqrt{1+\tan^2\psi}}$$

There are additional roots for $\psi = \psi_{above} \pm n\pi/2$, corresponding to relabelling the axes by rotating them through multiples of 90 degrees.

Finally, use these values to find the diagonal matrix elements:

I_{xx} = substitute $\sin\psi$ and $\cos\psi$ in xx element of I above

I_{yy} = substitute $\sin\psi$ and $\cos\psi$ in yy element of I above

$$I_{zz} = \frac{Mm}{M+m} \frac{1}{4} (2L^2 + h^2) + \frac{M}{12(L^2+2Lh)} (L^4 + L^3h + 8Lh^3)$$

With this choice of ψ, the diagonal elements are the principal moments.

What are the principal axes?

One lies in the symmetry plane, tilted about the c.m. point by an angle ψ toward the ball, starting from the direction parallel to the diagonal of the square face. The other also lies in the symmetry plane, tilted by ψ away from the ball, starting from the direction parallel to the short edges.

Rotated Inertia Tensor

$$\frac{m M}{4 (m + M)} \begin{pmatrix} h^2 \mathrm{Cos}[\psi]^2 - 2\sqrt{2}\, h L \mathrm{Cos}[\psi] \mathrm{Sin}[\psi] + 2 L^2 \mathrm{Sin}[\psi]^2 & -\sqrt{2}\, h L \mathrm{Cos}[2\psi] - (h^2 - 2 L^2) \mathrm{Cos}[\psi] \mathrm{Sin}[\psi] & 0 \\ -\sqrt{2}\, h L \mathrm{Cos}[2\psi] - (h^2 - 2 L^2) \mathrm{Cos}[\psi] \mathrm{Sin}[\psi] & 2 L^2 \mathrm{Cos}[\psi]^2 + h \mathrm{Sin}[\psi] \left(2\sqrt{2}\, L \mathrm{Cos}[\psi] + h \mathrm{Sin}[\psi]\right) & 0 \\ 0 & 0 & h^2 + 2 L^2 \end{pmatrix} +$$

$$\frac{M}{12 (2 h L + L^2)}$$

$$\begin{pmatrix} L (2 h^3 + 3 h^2 L + 4 h L^2 + L^3) \mathrm{Cos}[\psi]^2 + 2 L^3 (4 h + L) \mathrm{Sin}[\psi]^2 & L (-2 h^3 - 3 h^2 L + 4 h L^2 + L^3) \mathrm{Cos}[\psi] \mathrm{Sin}[\psi] & 0 \\ L (-2 h^3 - 3 h^2 L + 4 h L^2 + L^3) \mathrm{Cos}[\psi] \mathrm{Sin}[\psi] & 2 L^3 (4 h + L) \mathrm{Cos}[\psi]^2 + L (2 h^3 + 3 h^2 L + 4 h L^2 + L^3) \mathrm{Sin}[\psi]^2 & 0 \\ 0 & 0 & L (2 h^3 + 3 h^2 L + 4 h L^2 + L^3) \end{pmatrix}$$

In[20]:= $Izz = \frac{m M}{4 (m + M)} (h^2 + 2 L^2) + \frac{M}{12 (2 h L + L^2)} L (2 h^3 + 3 h^2 L + 4 h L^2 + L^3);$