solutions to

INERTIAL INTEGRALS

Worksheet 2

including worked examples

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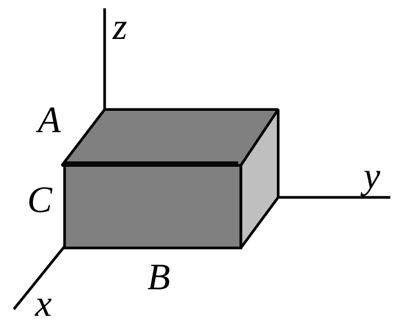
TOPICS page

The CageX inertial tensor in the center of mass will be addressed in the Worksheet on Translating Tensors.

A. SAMPLE CALCULATION: RECTANGULAR BRICK



A rectangular brick lies in the first octant of a Cartesian coordinate system with one corner at the origin.

Its lengths in the x, y, and z directions are A, B, and C respectively.

The brick has a uniform density ρ .

A.1. Center of mass. $\rho(x,y,z) \equiv$ mass per unit volume

The center of mass of a mass distribution $\rho(\boldsymbol{r})$ is defined as

$$\boldsymbol{R} \equiv \frac{1}{M} \int d^3r \ \rho(\boldsymbol{r}) \, \boldsymbol{r} \, .$$

The total mass M is the **norm** of the density distribution,

$$M = \int d^3r \ \rho(\boldsymbol{r}) \, .$$

The components of $\boldsymbol{R}$ are the normalized **first moments** of the density distribution $\rho(\boldsymbol{r})$,

$$X = \frac{1}{M} \int dz \int dy \int dx \ \ x \ \rho(x,y,z) \, ,$$

$$Y = \frac{1}{M} \int dz \int dy \int dx \ \ y \ \rho(x,y,z) \, ,$$

and

$$Z = \frac{1}{M} \int dz \int dy \int dx \ \ z \ \rho(x,y,z) \, .$$

The order of integrations doesn't matter for the result, and can be chosen for convenience.

The brick is a special case where the density ρ is constant inside a finite region.

$$\rho(x,y,z) = \rho \ \theta(x)\theta(A-x) \ \theta(y)\theta(B-y) \ \theta(z)\theta(C-z) \quad .$$

Find the total mass M and the center of mass $\boldsymbol{R}$ for the brick.

Solution

In the case of the brick, it is equally convenient to do the integrations in any order, because the range of integration for each variable is independent of the other variables.

$$M = \rho \times \text{volume} = \rho ABC \qquad \rho = \frac{ABC}{M}$$

Then $$X = \frac{1}{M}\int_0^C dz \int_0^B dy \int_0^A dx\; x\; \rho = \frac{1}{M}\, C\, B\, \tfrac{1}{2} A^2\, \rho \qquad X = \tfrac{1}{2} A$$

and $$Y = \frac{1}{M}\int_0^C dz \int_0^B dy\; y \int_0^A dx\; \rho = \frac{1}{M}\, C\, \tfrac{1}{2} B^2 A\; \rho \qquad Y = \tfrac{1}{2} B$$

and $$Z = \frac{1}{M}\int_0^C dz\; z \int_0^B dy \int_0^A dx\; \rho = \frac{1}{M}\, \tfrac{1}{2} C^2 A\, B\; \rho \qquad Z = \tfrac{1}{2} C$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \tfrac{1}{2} A \\ \tfrac{1}{2} B \\ \tfrac{1}{2} C \end{pmatrix}$$

check: the center of mass is at the center of the brick, as expected. √

A.2. Inertial tensor

The inertial tensor of a mass distribution $\rho(\boldsymbol{r})$ is defined as

$$I_{ij} = \int d^3r\, \rho(\boldsymbol{r})\, (\boldsymbol{r}^2 \delta_{ij} - r_i\, r_j\,).$$

To calculate the inertial tensor, it is convenient to first calculate the **second moments** M_{ij} , defined by

$$M_{ij} = \int d^3r\, \rho(\boldsymbol{r})\, r_i\, r_j\, .$$

Then the inertial tensor is

$$I_{ij} = -M_{ij} + \delta_{ij}\ \mathrm{tr}\, M = \begin{pmatrix} -M_{ij} \ \text{if } i \neq j \\ \sum_{k \neq i} M_{kk} \ \text{if } i = j \end{pmatrix}$$

The procedure for calculating M_{ij} is just like the procedure for calculating R_i , except for an extra factor of r_j in the integrand.

Then
$$M_{xx} = \int dz \int dy \int dx\ x^2\ \rho(x,y,z)\ ,$$

$$M_{yy} = \int dz \int dy \int dx\ y^2\ \rho(x,y,z)\ ,$$

$$M_{zz} = \int dz \int dy \int dx\ z^2\ \rho(x,y,z)\ ,$$

$$M_{xy} = M_{yx} = \int dz \int dy \int dx\ xy\ \rho(x,y,z)\ ,$$

$$M_{yz} = M_{zy} = \int dz \int dy \int dx\ yz\ \rho(x,y,z)\ ,$$

and
$$M_{xz} = M_{zx} = \int dz \int dy \int dx\ xz\ \rho(x,y,z)\ .$$

Solution

calculate $$M_{xx} = \int_0^A dx\ x^2 \int_0^B dy \int_0^C dz\ \rho = \frac{1}{3} A^3 B C \rho = \frac{1}{3} A^2 M$$

symmetry argument: interchanging $x \leftrightarrow y$ amounts to interchanging $A \leftrightarrow B$

$$M_{yy} = \frac{1}{3} B^2 M; \quad \text{similarly } M_{zz} = \frac{1}{3} C^2 M$$

calculate $$M_{xy} = \int_0^A dx\ x \int_0^B dy\ y \int_0^C dz\ \rho = \frac{A^2}{2} \frac{B^2}{2} C \rho = \frac{1}{4} AB\ M$$

check with symmetry argument: no change when interchanging $A \leftrightarrow B$ √

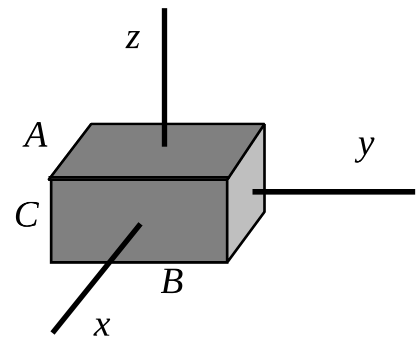
symmetry argument: interchanging $x \leftrightarrow z$ amounts to interchanging $A \leftrightarrow C$

$$M_{yz} = \frac{1}{4} BC\ M\ ; \quad \text{similarly } M_{xz} = \frac{1}{4} AC\ M$$

use these results for M_{ij} to construct the inertial tensor : $$I = \frac{M}{12} \begin{pmatrix} 4(B^2+C^2) & -3AB & -3AC \\ -3AB & 4(A^2+C^2) & -3BC \\ -3AC & -3BC & 4(A^2+B^2) \end{pmatrix}.$$

A.3. Inertial tensor with the origin at the center of mass of the box

A rectangular brick lies with its center at the origin of a Cartesian coordinate system, and with its edges parallel to the axes.



Its lengths in the *x*, *y*, and *z* directions are *A*, *B*, and *C* respectively.

The brick has a uniform density ρ .

Solution

The boundaries are now $x_{min} = -\frac{A}{2}$, $x_{max} = +\frac{A}{2}$, $y_{min} = -\frac{B}{2}$, $y_{max} = +\frac{B}{2}$,

$$z_{min} = -\frac{C}{2}, \; z_{max} = +\frac{C}{2} \; .$$

$$\text{Then } M_{xx} = \int_{-A/2}^{A/2} dx\, x^2 \int_{-B/2}^{B/2} dy \int_{-C/2}^{C/2} dz \;\; \rho \;=\; \left(\frac{1}{3}\left((A/2)^3 - (A/2)^3\right)\right) B\, C \;\; \rho$$

$$= \frac{1}{12} A^3 BC\, \rho = \frac{M}{12} A^2$$

$$A \leftrightarrow B \qquad M_{yy} = \frac{M}{12} B^2 , \quad \text{similarly } M_{zz} = \frac{M}{12} C^2 \; .$$

$$M_{xy} = \int_{-A/2}^{A/2} dx \;\; x \int_{-B/2}^{B/2} dy \;\; y \int_{-C/2}^{C/2} dz \;\; \rho = \left(\frac{1}{2}\left((A/2)^2 - (A/2)^2\right)\left((B/2)^2 - (B/2)^2\right)\right) C\, \rho = 0$$

$$\text{similarly, } M_{yz} = M_{xz} = 0.$$

Note: the vanishing of the off-diagonal elements reflects the symmetry of the mass distribution in this coordinate system.

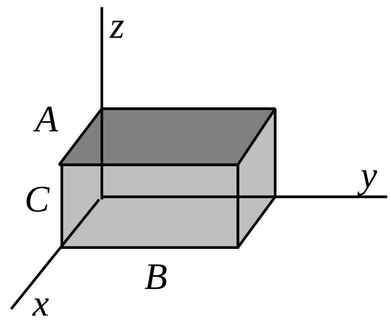
Use this M_{ij} to find

$$I_{\rm cm} = \frac{M}{12} \begin{pmatrix} B^2+C^2 & 0 & 0 \\ 0 & A^2+C^2 & 0 \\ 0 & 0 & A^2+B^2 \end{pmatrix} .$$

This result is related to the result of **A.2** above by the translation theorem for inertial tensors, as will be verified in the Worksheet *Translating Tensors*.

B. APPLICATION: HOLLOW RECTANGULAR CAGE

A hollow rectangular cage lies in the first octant of a Cartesian coordinate system with one corner at the origin.



Its lengths in the x, y, and z directions are A, B, and C respectively.

The walls of the cage have a uniform surface mass density σ .

The total mass of the cage is the sum of the masses of its walls.

The top and bottom walls each have area AB and mass $AB\sigma$.

The front and back walls each have area BC and mass $BC\sigma$.

The left and right walls each have area AC and mass $AC\sigma$.

The total mass is $$M = 2(AB + BC + AC)\sigma$$

The mass density of each wall is

$$\rho_{top}(x,y,z) = \sigma\ \delta(z-C)\ \theta(x)\ \theta(A-x)\ \ \theta(y)\ \theta(B-y)\ ,$$

$$\rho_{bottom}(x,y,z) = \sigma\ \delta(z)\ \theta(x)\ \theta(A-x)\ \ \theta(y)\ \theta(B-y)\ ,$$

$$\rho_{front}(x,y,z) = \sigma\ \delta(x-A)\ \theta(y)\ \theta(B-y)\ \ \theta(z)\ \theta(C-z)\ ,$$

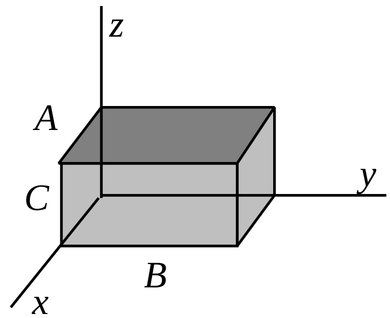
$$\rho_{back}(x,y,z) = \sigma\ \delta(x)\ \theta(y)\ \theta(B-y)\ \ \theta(z)\ \theta(C-z)\ ,$$

$$\rho_{right}(x,y,z) = \sigma\ \delta(y-B)\ \theta(x)\ \theta(A-x)\ \ \theta(z)\ \theta(C-z)\ ,$$

$$\rho_{left}(x,y,z) = \sigma\ \delta(y)\ \theta(x)\ \theta(A-x)\ \ \theta(z)\ \theta(C-z)\ .$$

In each case, the θ funcitons limit the ranges of integration for two of the variables, and the third integration uses the delta function.

B.1. Center of mass.



We can find the contributions of each wall to the center-of-mass coordinate X :

front: $$\frac{1}{M}\int_0^C dz \int_0^B dy \int_{-\infty}^{\infty} dx\ x\ \sigma\ \delta(x-A) = \frac{1}{M}\ C\ B\ A\ \sigma$$

back: $$\frac{1}{M}\int_0^C dz \int_0^B dy \int_{-\infty}^{\infty} dx\ x\ \sigma\ \delta(x) = 0$$

top: $$\frac{1}{M}\int_{-\infty}^{\infty} dz \int_0^B dy \int_0^A dx\ x\ \sigma\ \delta(z-C) = \frac{1}{M}\ B\ \tfrac{1}{2}A^2\ \sigma$$

bottom: $$\frac{1}{M}\int_{-\infty}^{\infty} dz \int_0^B dy \int_0^A dx\ x\ \sigma\ \delta(z) = \frac{1}{M}\ B\ \tfrac{1}{2}A^2\ \sigma$$

right: $$\frac{1}{M}\int_0^C dz \int_{-\infty}^{\infty} dy \int_0^A dx\ x\ \sigma\ \delta(y-B) = \frac{1}{M}\ C\ \tfrac{1}{2}A^2\ \sigma$$

left: $$\frac{1}{M}\int_0^C dz \int_{-\infty}^{\infty} dy \int_0^A dx\ x\ \sigma\ \delta(y) = \frac{1}{M}\ C\ \tfrac{1}{2}A^2\ \sigma$$

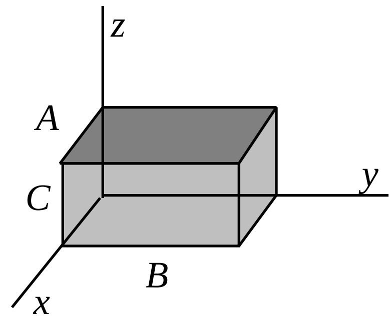
total: $$X = \frac{\sigma}{M}\ A\ (BC + AB + AC) = \frac{\sigma}{2\sigma(AB+BC+AC)}\ A(BC+AB+AC) = \frac{A}{2}$$

We can obtain similar results for Y and Z by repeating the computation, or by permuting A, B ,and C appropriately:

$$A \leftrightarrow B \qquad Y = \frac{B}{2}\ , \qquad A \leftrightarrow C \qquad Z = \frac{C}{2}$$

Comment: no surprises here! But we confirm our computational method. √

B.2. Inertial tensor for the rectangular cage



We calculate the contributions of each face to the moments M_{xx} and M_{sy} . The other moments can be obtained by permutations.

First, contributions to M_{xx} :

front: $$\int_0^C dz \int_0^B dy \int_{-\infty}^{\infty} dx\; x^2\, \sigma\, \delta(x-A) = C\, B\, A^2\, \sigma$$

back: $$\int_0^C dz \int_0^B dy \int_{-\infty}^{\infty} dx\; x^2\, \sigma\, \delta(x) = 0$$

top: $$\int_{-\infty}^{\infty} dz \int_0^B dy \int_0^A dx\; x^2\, \sigma\, \delta(z-C) = B\, \tfrac{1}{3}A^3\, \sigma$$

bottom: $$\int_{-\infty}^{\infty} dz \int_0^B dy \int_0^A dx\; x^2\, \sigma\, \delta(z) = B\, \tfrac{1}{3}A^3\, \sigma$$

right: $$\int_0^C dz \int_{-\infty}^{\infty} dy \int_0^A dx\; x^2\, \sigma\, \delta(y-B) = C\, \tfrac{1}{3}A^3\, \sigma$$

left: $$\int_0^C dz \int_{-\infty}^{\infty} dy \int_0^A dx\; x^2\, \sigma\, \delta(y) = C\, \tfrac{1}{3}A^3\, \sigma$$

total: $$M_{xx} = \sigma \frac{A^2}{3} (3BC + 2AB + 2AC) = \tfrac{1}{3} MA^2 + \frac{\sigma}{3} A^2 B\, C$$

We can obtain similar results for Y and Z by repeating the computation, or by permuting A, B ,and C appropriately:

$$x \leftrightarrow y \qquad A \leftrightarrow B : M_{yy} = \sigma \frac{B^2}{3} (3AC + 2AB + 2BC) = \tfrac{1}{3} MB^2 + \frac{\sigma}{3} A\, B^2 C$$

$$x \leftrightarrow z \qquad A \leftrightarrow C : M_{zz} = \sigma \frac{C^2}{3} (3AB + 2AC + 2BC) = \tfrac{1}{3} MC^2 + \frac{\sigma}{3} A\, B\, C^2$$

B.2. Inertial tensor for the rectangular cage (continued)

Next, contributions to M_{xy} :

front: $$\int_0^C dz \int_0^B dy\; y \int_{-\infty}^{\infty} dx\; x\; \sigma\, \delta(x-A) = C \; \tfrac{1}{2}B^2 \; A \; \sigma$$

back: $$\int_0^C dz \int_0^B dy\; y \int_{-\infty}^{\infty} dx\; x\; \sigma\, \delta(x) = 0$$

top: $$\int_{-\infty}^{\infty} dz \int_0^B dy\; y \int_0^A dx\; x\; \sigma\, \delta(z-C) = \tfrac{1}{2}B^2 \; \tfrac{1}{2}A^2 \; \sigma$$

bottom: $$\int_{-\infty}^{\infty} dz \int_0^B dy\; y \int_0^A dx\; x\; \sigma\, \delta(z) = \tfrac{1}{2}B^2 \; \tfrac{1}{2}A^2 \; \sigma$$

right: $$\int_0^C dz \int_{-\infty}^{\infty} dy\; y \int_0^A dx\; x\; \sigma\, \delta(y-B) = C\, B \; \tfrac{1}{2}A^2 \; \sigma$$

left: $$\int_0^C dz \int_{-\infty}^{\infty} dy\; y \int_0^A dx\; x\; \sigma\, \delta(y) = 0$$

total: $$M_{xy} = \sigma \frac{AB}{4} \left(2BC + 2AB + 2AC\right) = \frac{AB}{4} M$$

We can obtain similar results for M_{xz} and M_{yz} by repeating the computation, or by permuting A, B ,and C appropriately:

$$z \leftrightarrow y \qquad C \leftrightarrow B \quad : M_{xz} = \frac{AC}{4} M$$

further permute this result: $x \leftrightarrow y \qquad A \leftrightarrow B \quad : M_{yz} = \dfrac{BC}{4} M$

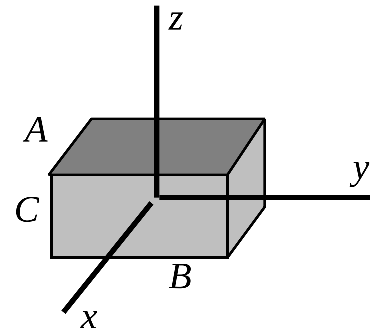
$$I = \frac{\sigma}{12} \begin{pmatrix} 12AB^2C+8B^3(A+C) + 12ABC^2+8C^3(A+B) & -6AB(AB+BC+AC) & -6AC(AB+BC+AC) \\ -6AB(AB+BC+AC) & 12A^2BC+8A^3(B+C) + 12ABC^2+8C^3(A+B) & -6BC(AB+BC+AC) \\ -6AC(AB+BC+AC) & -6BC(AB+BC+AC) & 12A^2BC+8A^3(B+C) + 12AB^2C+8B^3(A+C) \end{pmatrix}$$

B.3. Inertial tensor in the center of mass for the rectangular cage

A hollow rectangular cage centered on the origin of a Cartesian coordinate system with sides parallel to the axes.

Its lengths in the x, y, and z directions are A, B, and C respectively.

The walls of the cage have a uniform surface mass density σ .



to find M_{xx} :

front: $$\int_{-C/2}^{C/2} dz \int_{-B/2}^{B/2} dy \int_{-\infty}^{\infty} dx \; x^2 \; \sigma \, \delta(x-A/2) = C\,B \left(\frac{A}{2}\right)^2 \sigma$$

back: $$\int_{-C/2}^{C/2} dz \int_{-B/2}^{B/2} dy \int_{-\infty}^{\infty} dx \; x^2 \; \sigma \, \delta(x+A/2) = C\,B \left(\frac{A}{2}\right)^2 \sigma$$

top: $$\int_{-\infty}^{\infty} dz \int_{-B/2}^{B/2} dy \int_{-A/2}^{A/2} dx \; x^2 \; \sigma \, \delta(z-C/2) = B \; \frac{2}{3}\left(\frac{A}{2}\right)^3 \sigma$$

bottom: $$\int_{-\infty}^{\infty} dz \int_{-B/2}^{B/2} dy \int_{-A/2}^{A/2} dx \; x^2 \; \sigma \, \delta(z+C/2) = B \; \frac{2}{3}\left(\frac{A}{2}\right)^3 \sigma$$

right: $$\int_{-C/2}^{C/2} dz \int_{-\infty}^{\infty} dy \int_{-A/2}^{A/2} dx \; x^2 \; \sigma \, \delta(y-B/2) = C \; \frac{2}{3}\left(\frac{A}{2}\right)^3 \sigma$$

left: $$\int_{-C/2}^{C/2} dz \int_{-\infty}^{\infty} dy \int_{-A/2}^{A/2} dx \; x^2 \; \sigma \, \delta(y+B/2) = C \; \frac{2}{3}\left(\frac{A}{2}\right)^3 \sigma$$

total: $$M_{xx} = \sigma \frac{A^2}{6} (3BC + AB + AC)$$

permuting:

$x \leftrightarrow y \quad A \leftrightarrow B : M_{yy} = \sigma \frac{B^2}{6} (3AC + AB + BC)$

$x \leftrightarrow z \quad A \leftrightarrow C : M_{zz} = \sigma \frac{C^2}{6} (3AB + AC + BC)$

B.3. Inertial tensor in the center of mass for the rectangular cage

(continued)

To find M_{xy} :

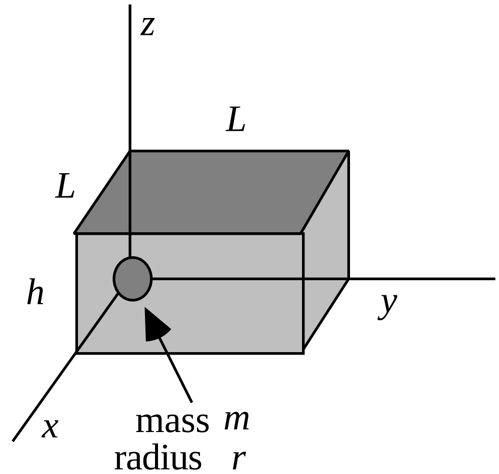
front: $$\int_{-C/2}^{C/2} dz \int_{-B/2}^{B/2} dy\, y \int_{-\infty}^{\infty} dx\, x\, \sigma\, \delta(x-A/2) = 0 .$$

Indeed, every off-diagonal moment vanishes, because the integrands are odd, while the ranges of integration are even.

result:

$$I = \frac{\sigma}{6} \begin{pmatrix} B^3(A+C)+3AB^2C + C^3(A+B)+3ABC^2 & 0 & 0 \\ 0 & A^3(B+C)+3A^2BC + C^3(A+B)+3ABC^2 & 0 \\ 0 & 0 & A^3(B+C)+3A^2BC + B^3(A+C)+3AB^2C \end{pmatrix}$$

C. APPLICATION: CAGEX APPARATUS



The CageX apparatus consists of a hollow rectangular cage with two square faces and four narrow rectangular faces.

Its length in the x and y directions is L, and the length in the z direction is h .

The walls of the cage have a uniform surface mass density, and its total mass is M .

In addition there is a clay sphere of mass m fastened to the corner of the cage at the origin. The radius of this sphere is r .

1. Center of mass for CageX apparatus

The expression in terms of M , m , L , h , and r is:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \frac{M}{M+m} \begin{pmatrix} \frac{1}{2} L \\ \frac{1}{2} L \\ \frac{1}{2} h \end{pmatrix}.$$

2. Inertial tensor for CageX apparatus

The expression for the inertial tensor of the **ball** in terms of r and m is approximately

$$I_{ball} = \frac{2}{5} m r^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From **B.2** above the expression for the inertial tensor of the **cage** in terms of M, L, and h is approximately

$$I_{cage} = \frac{M}{24(L^2+2Lh)} \begin{pmatrix} 8L^4+20L^3h+12L^2h^2+16Lh^3 & -6L^3(L+2h) & -6L^2h(L+2h) \\ -6L^3(L+2h) & 8L^4+20L^3h+12L^2h^2+16Lh^3 & -6L^2h(L+2h) \\ -6L^2h(L+2h) & -6L^2h(L+2h) & 16L^4+40L^3h \end{pmatrix}$$

3. Numerical values

The numerical coordinates of the center of mass are

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \frac{M}{M+m} \begin{pmatrix} \frac{1}{2}L \\ \frac{1}{2}L \\ \frac{1}{2}h \end{pmatrix} = \begin{pmatrix} \_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_ \end{pmatrix}$$

inertial tensor of **ball** inertial tensor of **cage** *units!*

$$\begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}, \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$$

We can see that the contribution of the ball to the inertial tensor (with respect to the given origin) is ___negligibly small______________ .