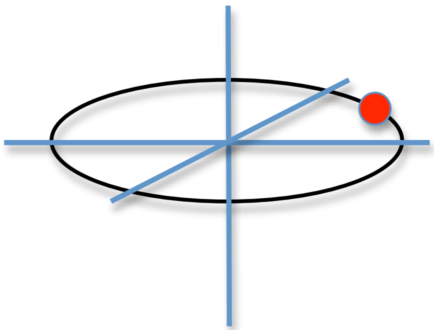
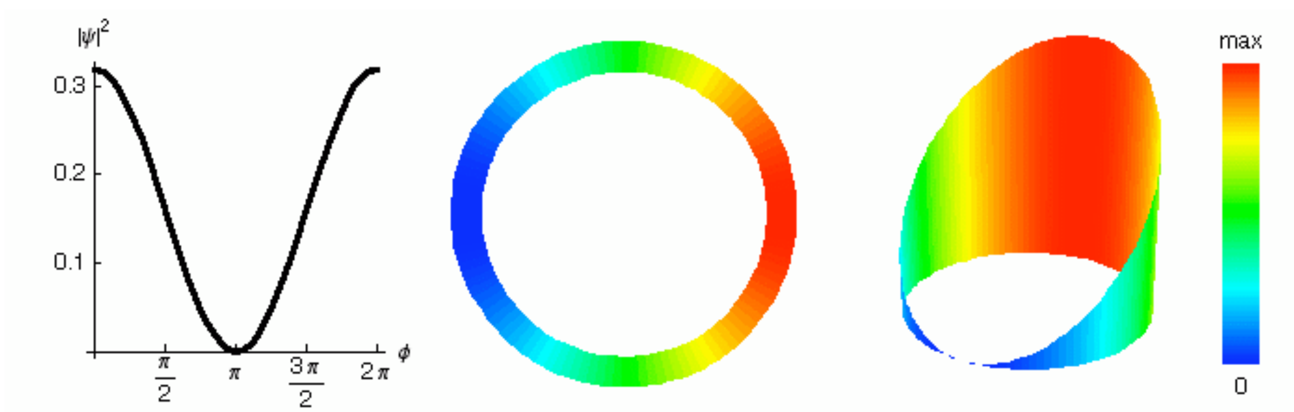


$$\frac{L_z^2}{2I}\left(\frac{1}{\sqrt{2\pi}}e^{im\phi}\right)=\frac{m^2\hbar^2}{2I}\left(\frac{1}{\sqrt{2\pi}}e^{im\phi}\right)$$

Toy problem: a quantum particle confined to a ring

Reading: McIntyre 7.5



Separation of variables:

- Blue has angular dependence, red is radial:

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{1}{\hbar^2 r^2}\mathbf{L}^2\right]R(r)Y(\theta,\phi)+V(r)R(r)Y(\theta,\phi)$$

$$=ER(r)Y(\theta,\phi)$$

Algebra (follow book 7.4):

$$\underbrace{\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)-\frac{2\mu}{\hbar^2}(E-V(r))r^2}_{function\ \ of\ \ r\ \ only}=\underbrace{\frac{1}{\hbar^2}\frac{1}{Y(\theta,\phi)}\mathbf{L}^2Y(\theta,\phi)}_{function\ \ of\ \ \theta,\phi\ \ only}\equiv A$$

Separation of variables:

- Once we solve the (blue) angular problem, it is the solution to the angular part of ALL central force problems!

$$\frac{1}{\hbar^2}\mathbf{L}^2 Y(\theta,\phi) = A Y(\theta,\phi)$$

- Once we find *A* (and Y), plug back into red equation and solve to find *E* (and *R*(*r*)).

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - \frac{2\mu}{\hbar^2}(E - V(r))r^2 R(r) \equiv A R(r)$$

Separate angular equation

- But we need to work on blue equation more, first.

$$\mathbf{L}^2 \doteq -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$$

$$\left[\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right) - B \frac{1}{\sin^2\theta} \right] \Theta(\theta) = -A\Theta(\theta)$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} = -B\Phi(\phi)$$

Summary

- Here's the plan:

$$\frac{d^2\Phi(\phi)}{d\phi^2} = -B\Phi(\phi)$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - B\frac{1}{\sin^2\theta}\right]\Theta(\theta) = -A\Theta(\theta)$$

$$\left[-\frac{\hbar^2}{2\mu r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + V(r) + A\frac{\hbar^2}{2\mu r^2}\right]R(r) = ER(r)$$

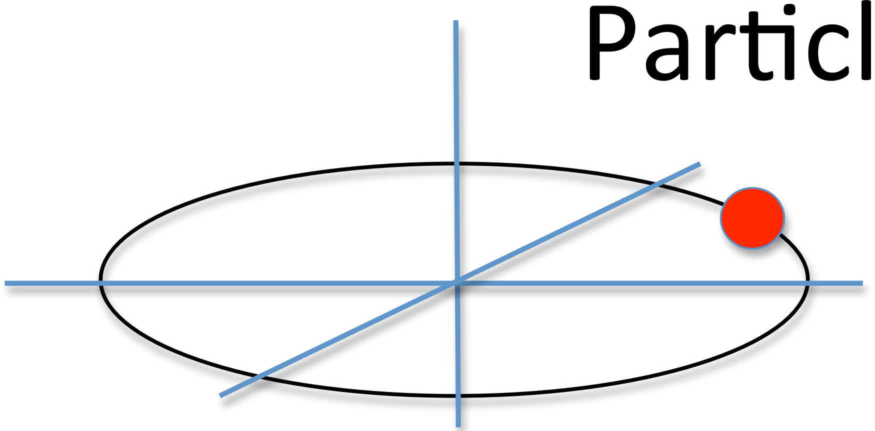
$$\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)\Theta_\ell^m(\theta)\Phi_m(\phi)$$

$$\psi(r,\theta,\phi,t) = \sum_{n\ell m} c_{n\ell m} R_{n\ell}(r)\Theta_\ell^m(\theta)\Phi_m(\phi)e^{-iE_n t/\hbar}$$

Summary

- Here's the plan:
- We'll consider 3 different systems, a ring (to solve the ϕ problem), a sphere (to solve the θ and ϕ problem), and the full hydrogen atom (to solve the (r, θ, ϕ problem)
- We'll find the quantum numbers and wave functions that solve each problem
- We'll apply all the things you've learned in PH424 and PH425
- **Please read ahead** – the math is much more intense (though not harder) than before

Particle on a ring



$$\mathbf{r} = r_0 \cos\phi \mathbf{i} + r_0 \sin\phi \mathbf{j}$$

$$\theta = \pi / 2$$

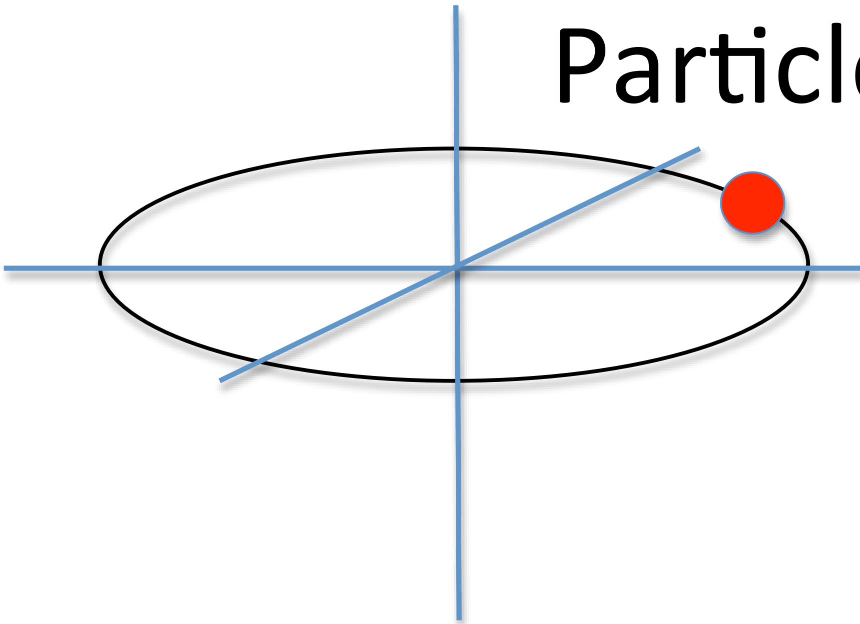
$$H_{ring} \left| E_{ring} \right\rangle = E_{ring} \left| E_{ring} \right\rangle$$

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi)$$

$$+V(r)\psi(r,\theta,\phi) = E_{ring}\psi(r,\theta,\phi)$$

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r_0^2}\frac{\partial^2}{\partial\phi^2}\right]\psi(\phi) + \underbrace{V(r_0)}_{assume\ =\ 0}\psi(\phi) = E_{ring}\psi(\phi)$$

Particle on a ring



$$I = \mu r_0^2$$

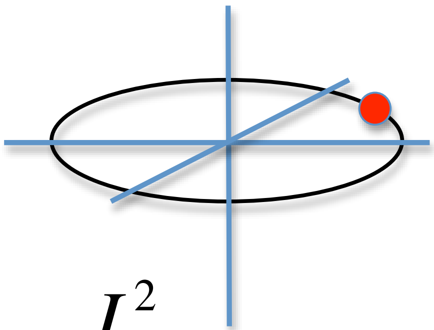
$$L_z \doteq -i\hbar \frac{\partial}{\partial \phi}$$

$$-\frac{\hbar^2}{2I}\frac{d^2}{d\phi^2}\Phi(\phi) = E_{ring}\Phi(\phi) \xrightarrow{\text{Looks like}} \frac{d^2\Phi(\phi)}{d\phi^2} = -B\Phi(\phi)$$

$$\frac{L_z^2}{2I}\Phi(\phi) = E_{ring}\Phi(\phi)$$

$$B = \frac{2I}{\hbar^2}E_{ring}$$

$$\longrightarrow H_{ring} = \frac{L_z^2}{2I}$$



Particle on a ring

$$\frac{L_z^2}{2I}\Phi(\phi) = E_{ring}\Phi(\phi)$$

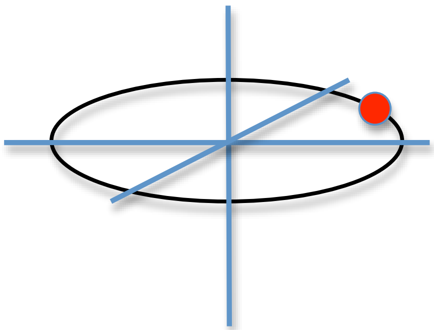
$$L_z|m\rangle = m\hbar|m\rangle$$

$$L_z^2|m\rangle = m^2\hbar^2|m\rangle$$

$$\frac{L_z^2}{2I}|m\rangle = \frac{m^2\hbar^2}{2I}|m\rangle$$

$$H_{ring}|m\rangle = E_{ring}|m\rangle \quad \Rightarrow \quad E_{ring} = \frac{m^2\hbar^2}{2I}$$

$$B = \frac{2I}{\hbar^2}E_{ring} = m^2$$



Particle on a ring

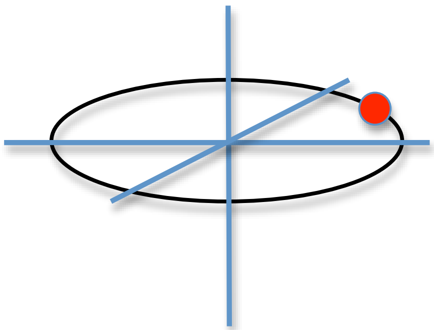
Spatial Solution

$$\frac{d^2\Phi(\phi)}{d\phi^2} = -B\Phi(\phi)$$

$$\Phi(\phi) = Ne^{\pm i\sqrt{B}\phi}$$

$$\Phi(\phi + 2\pi) = \Phi(\phi) \quad \Rightarrow \quad \sqrt{B} = m = 0,\ \pm1,\ \pm2,\ \ldots$$

$$|m\rangle \doteq \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$



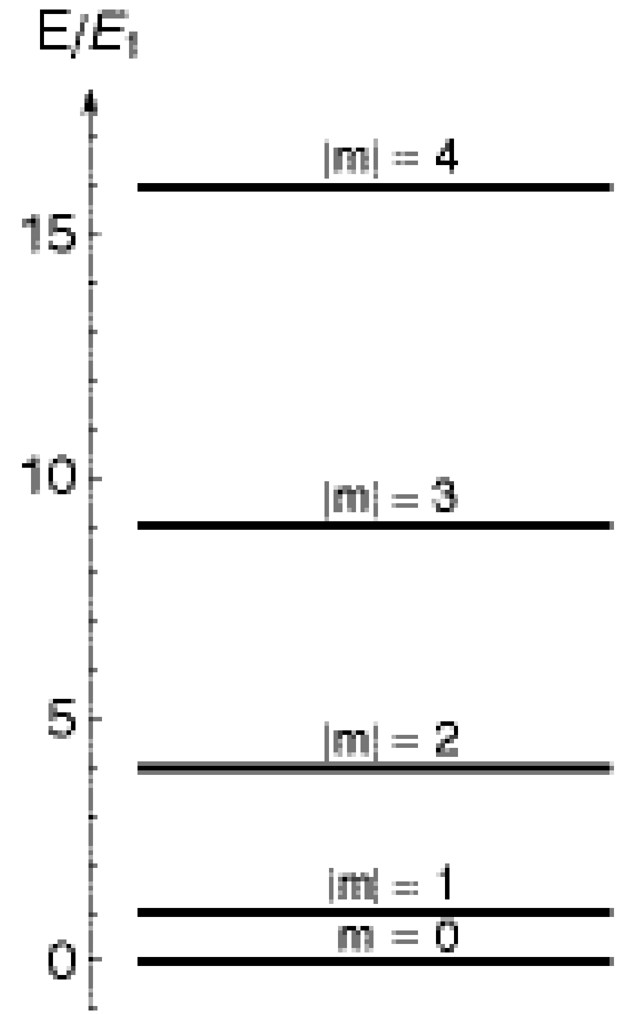
Particle on a ring

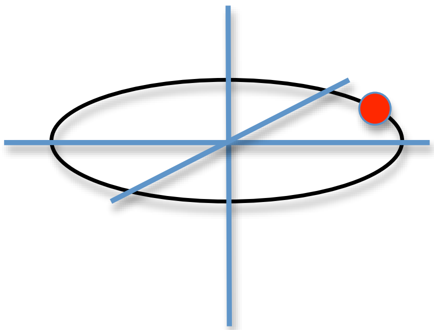
$$H_{ring}\left|m\right\rangle = E_{ring}\left|m\right\rangle$$

$$H_{ring}\Phi_m\left(\phi\right) = E_{ring}\Phi_m\left(\phi\right)$$

$$E_{|m|} = m^2\frac{\hbar^2}{2I}$$

$$\left|m\right\rangle \doteq \Phi_m\left(\phi\right) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$



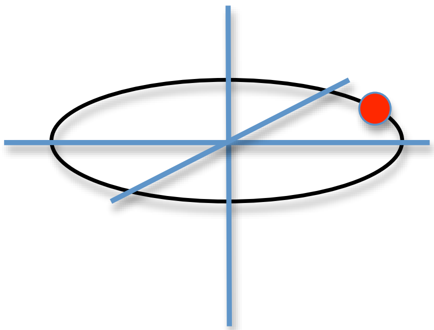


Particle on a ring

m = 0, ±1, ±2 … are allowed – what are boundary conditions that determine this?

How is this similar to and different than the infinite square well problem?

m = 1 and m = -1 (and ±2 *etc.*) correspond to SAME energy : DEGENERACY



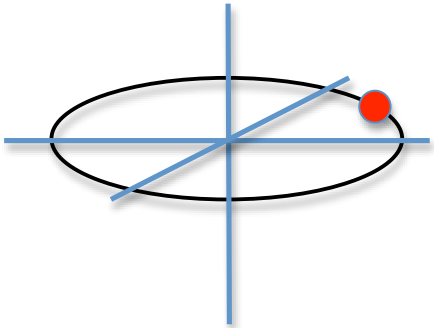
Particle on a ring

Connect to bra-ket notation ….

$$H\left|m\right\rangle = E\left|m\right\rangle \rightarrow \frac{L_z^2}{2I}\left|m\right\rangle = \frac{m^2\hbar^2}{2I}\left|m\right\rangle$$

The ket can also be represented as a column vector

$$\left|m=1\right\rangle \doteq \begin{pmatrix} \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \begin{matrix} \leftarrow 2 \\ \leftarrow m=1 \\ \leftarrow 0 \\ \leftarrow -1 \\ \leftarrow -2 \end{matrix} \qquad L_z \doteq \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \unicode{x22F0} \\ \cdots & +1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & -1 & \cdots \\ \unicode{x22F0} & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \leftarrow +2 \\ \leftarrow m=+1 \\ \leftarrow 0 \\ \leftarrow -1 \\ \leftarrow -2 \end{matrix}$$



Particle on a ring

L_z and H commute …

$$H\left|m\right\rangle = \frac{L_z^2}{2I}\left|m\right\rangle = \frac{m^2\hbar^2}{2I}\left|m\right\rangle \qquad L_z\left|m\right\rangle = m\hbar\left|m\right\rangle$$

Superposition states yield different energies, angular momenta; practice this in worksheets.

$$\left|\psi\right\rangle = \frac{1}{2}\left[\left|1\right\rangle + i\left|-2\right\rangle + e^{i\delta}\left|0\right\rangle + \left|-1\right\rangle\right]$$

$$\psi(\phi) = \frac{1}{2}\left[\frac{e^{i\phi}}{\sqrt{2\pi}} + i\frac{e^{-i2\phi}}{\sqrt{2\pi}} + e^{i\delta}\frac{1}{\sqrt{2\pi}} + \frac{e^{-i\phi}}{\sqrt{2\pi}}\right]$$

As column vector?

Summary

- This is a "toy" system, but some aspects are instructive
- The Hamiltonian is proportional to L_z^2
- The eigenstates of L_z and H (L_z^2) are the same because the two operators commute.
- In the (position) wave function representation, the eigenstates are $(1/2^{1/2})$* exp($im\phi$)
- In bra-ket notation, we denote the state by the quantum number *m*, that can be 0,±1…
- In matrix representation, the eigenstates are column vectors (infinite!) with one entry
- The eigenvalues are $mh/2\pi$ for L_z and $m^2h^2/2I(2\pi)^2$ for H
  (be warned that this H goes like L^2, not L_z^2 for real problems)
- There is degeneracy in this system; more than one eigenstate corresponds to the same energy