L4 – Equations of motion

Read course notes section 9, 11

Taylor 8.3

Newton's Law for fictitious reduced mass

$$f(r)\hat{r} = \mu \frac{d^2\vec{r}}{dt^2}$$

$$= \mu\left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \mu\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

$$\frac{d}{dt}\left(\mu r^2\dot{\phi}\right) = 0 \Rightarrow \mu r^2\dot{\phi} \equiv \ell = \text{constant}$$

$$\ddot{r} = \frac{\ell^2}{\mu^2 r^3} + \frac{1}{\mu} f(r) \Rightarrow r(t) \qquad\qquad r(t) \Rightarrow \phi(t)$$

Newton's Law for fictitious reduced mass

$$\frac{d^2r}{dt^2} = \frac{\ell^2}{\mu^2 r^3} + \frac{1}{\mu} f(r) \qquad \frac{d\phi}{dt} = \frac{\ell}{\mu r^2}$$

- Cut out the middle man and go right for *r*(ϕ)!
- Use u=1/r

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \dot{\phi}\frac{dr}{d\phi} = \frac{\ell}{\mu r^2}\frac{dr}{d\phi} = -\frac{\ell}{\mu}\frac{d(1/r)}{d\phi} = -\frac{\ell}{\mu}\frac{du}{d\phi}$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-\frac{\ell}{\mu}\frac{du}{d\phi}\right) = -\frac{\ell}{\mu}\frac{d^2u}{d\phi^2}\dot{\phi} = -\frac{\ell^2}{\mu^2}u^2\frac{d^2u}{d\phi^2}$$

Orbit for gravitational force

$$\frac{d^2 r}{dt^2} = \frac{\ell^2}{\mu^2 r^3} + \frac{1}{\mu} f(r)$$

$$-\frac{\ell^2}{\mu^2} u^2 \frac{d^2 u}{d\phi^2} - \frac{\ell^2}{\mu^2} u^3 = \frac{1}{\mu} f\left(\tfrac{1}{u}\right)$$

$$\frac{d^2 u}{d\phi^2} + u = -\frac{\mu}{\ell^2 u^2} f\left(\tfrac{1}{u}\right) \Rightarrow \frac{d^2 u}{d\phi^2} + u = \frac{\mu k}{\ell^2}$$

- What is this equation?

Orbit for gravitational force

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu k}{\ell^2}$$

- Like a driven harmonic oscillator (PH 421!)
- Solution is sum of
  homogeneous solution …. which is ?
  and
  particular solution …. which is?

Orbit for gravitational force

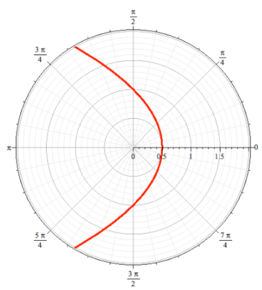
$$\frac{d^2u}{d\phi^2}+u=\frac{\mu k}{\ell^2}$$

- homogeneous solution $\frac{d^2u}{d\phi^2}+u=0 \Rightarrow u=\frac{1}{r}=A\sin\phi+B\cos\phi$

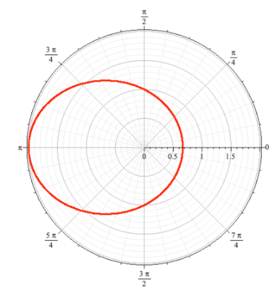
- particular solution $u=\frac{1}{r}=\frac{\mu k}{\ell^2}$

$$r=\frac{1}{A\sin\phi+B\cos\phi+\frac{\mu k}{\ell^2}}$$

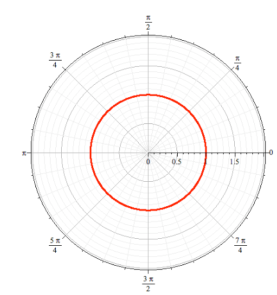
Orbit for gravitational force

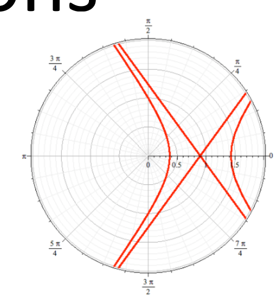


$$r(\phi) = \frac{\dfrac{\ell^2}{\mu k}}{1 + \varepsilon \cos(\phi + \delta)}$$



- radius of orbit set by ang. mom, μ, and Gm_1m_2
- Shape set by initial conditions and ang. Mom., μ, Gm_1m_2

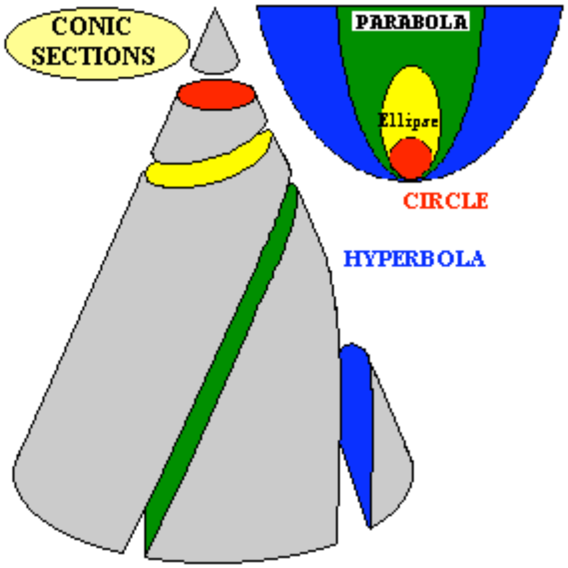


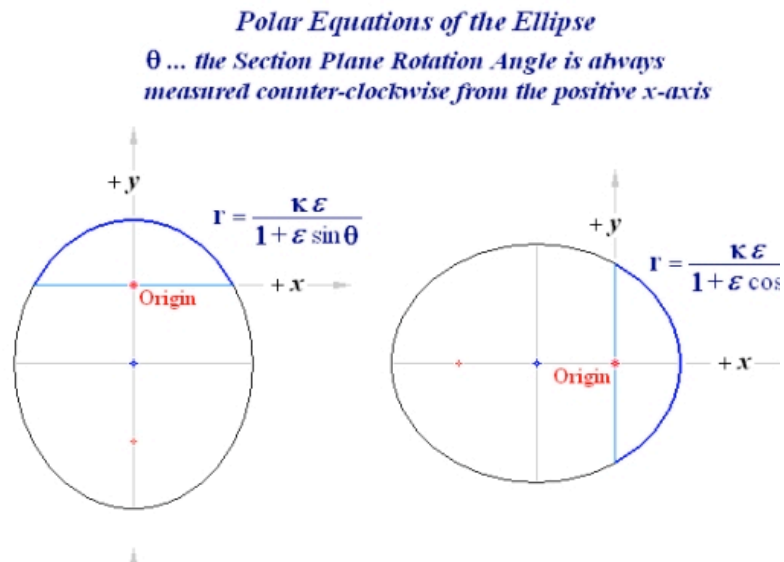


Look at orbits

PhET simulation

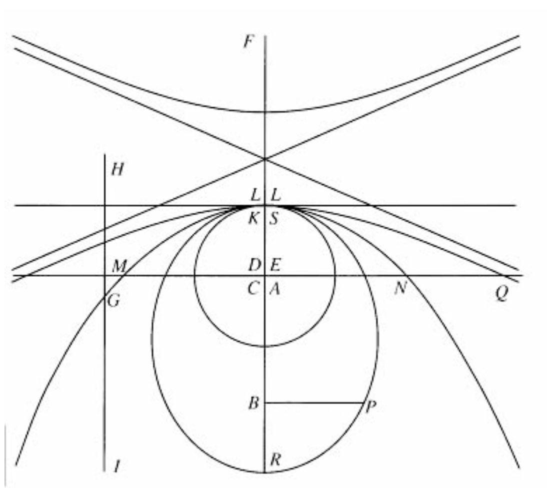
Orbits.jar -> shows orbits with eff pot

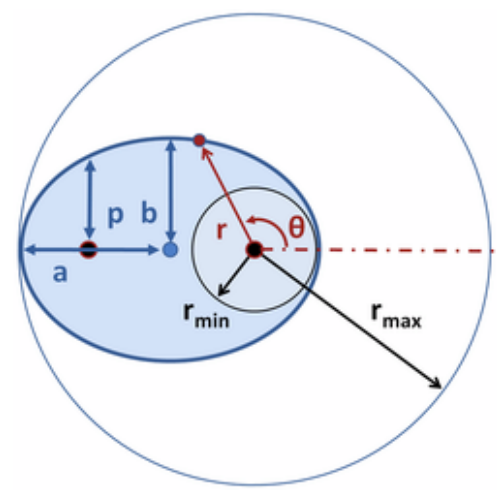




Conic sections

Reading: Course packet



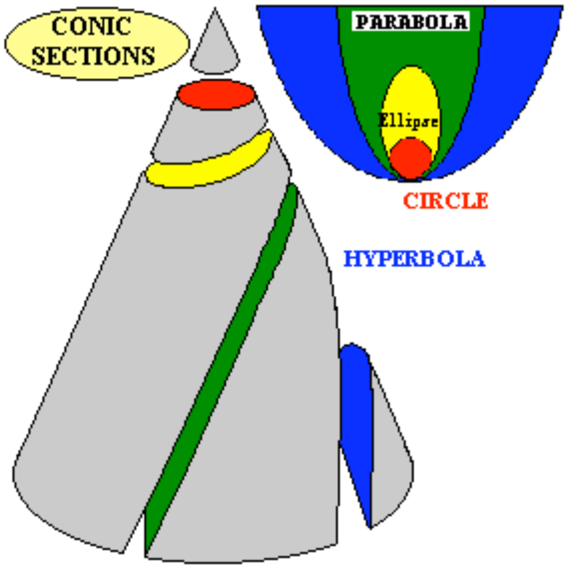


Explore this conic section:

$$r(\phi) = \frac{\alpha}{1 + \varepsilon \cos\phi}$$

What do α and ε represent?
Explore special values of epsilon
What are the shapes called?
Can you "tip" them?

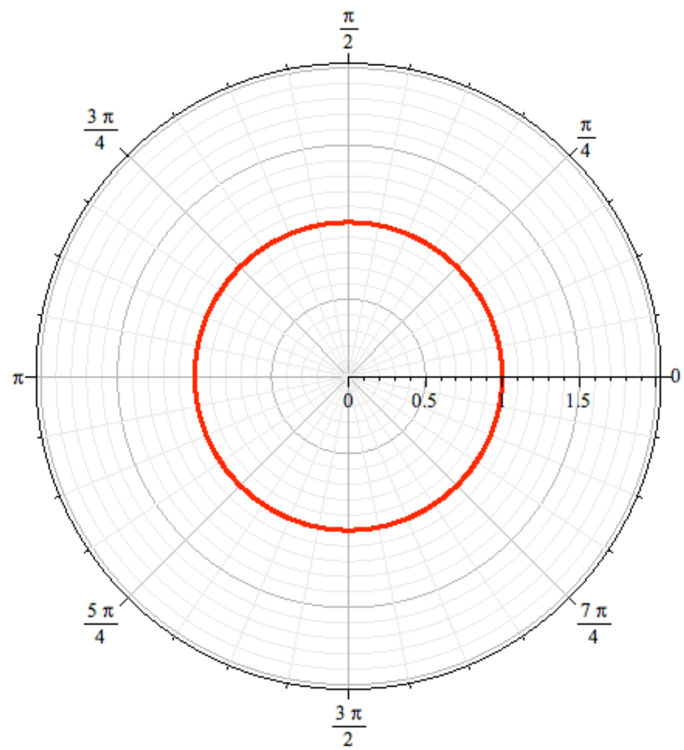
This form will be the solution of the equation of motion for a particle of (reduced) mass μ in a central gravitational field

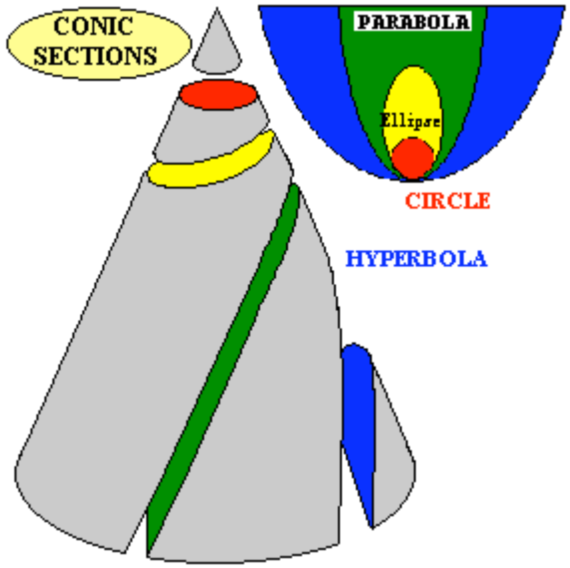


Circles: ε = 0

$$r(\phi) = \frac{\alpha}{1 + \varepsilon \cos\phi}$$

α sets the scale
ε determines the shape



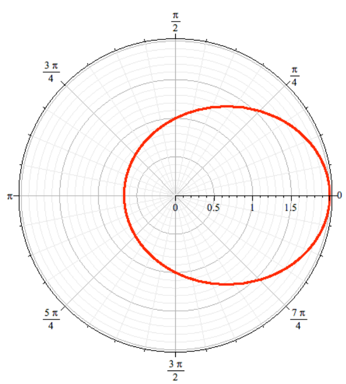


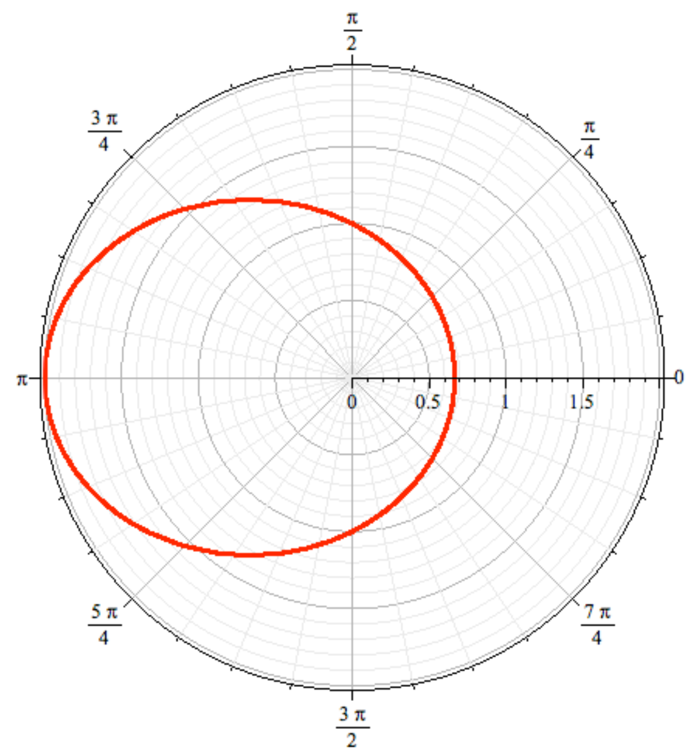
Ellipses: $0 < \varepsilon < 1$

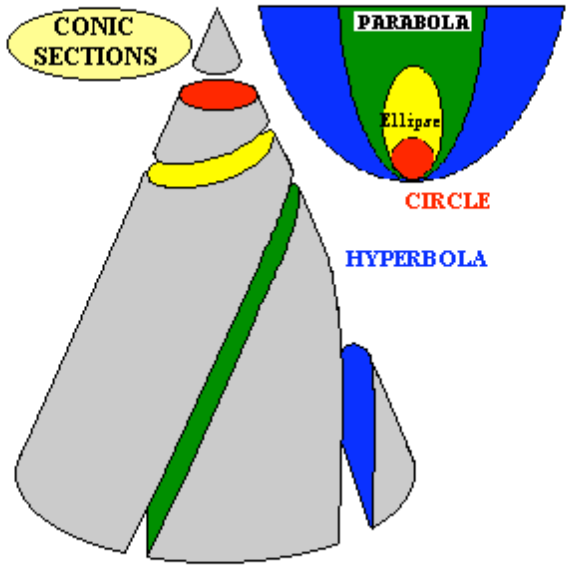
$$r(\phi) = \frac{\alpha}{1 + \varepsilon \cos\phi}$$

α sets the scale

ε determines the shape



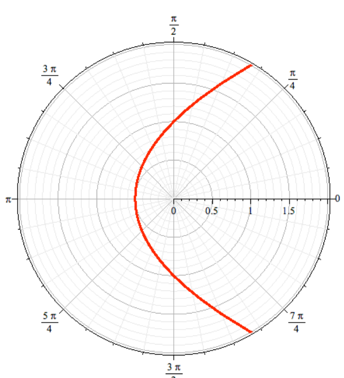


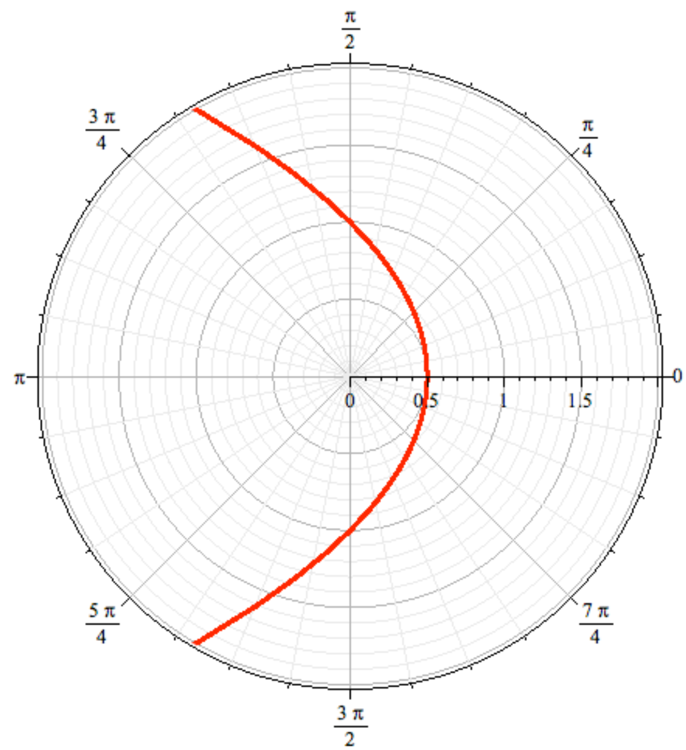


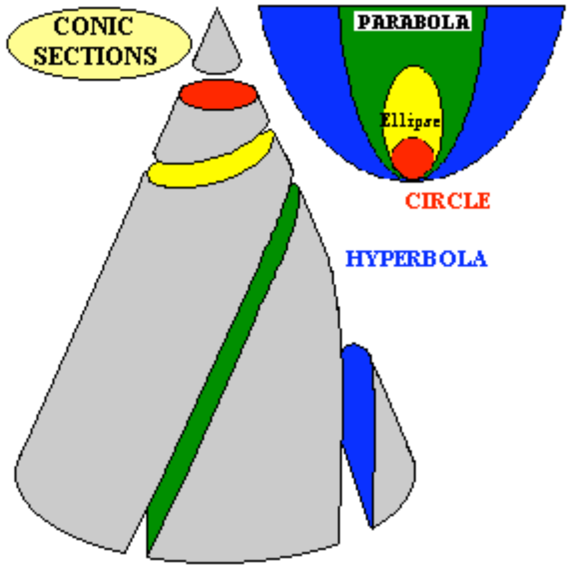
Parabola: ε = 1

$$r(\phi) = \frac{\alpha}{1 + \varepsilon\cos\phi}$$

α sets the scale
ε determines the shape



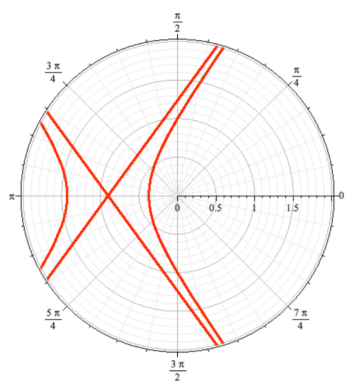


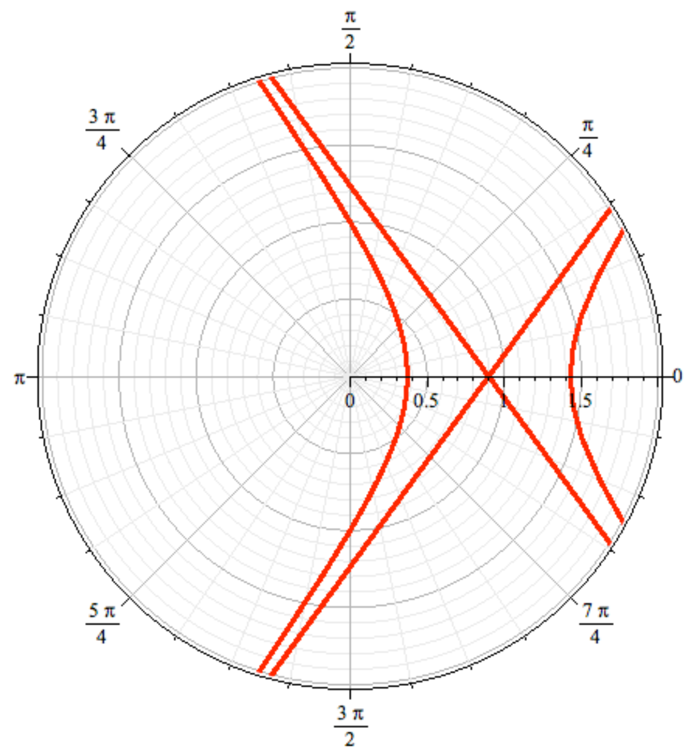


Hyperbola: ε > 1

$$r(\phi) = \frac{\alpha}{1 + \varepsilon \cos\phi}$$

α sets the scale
ε determines the shape





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