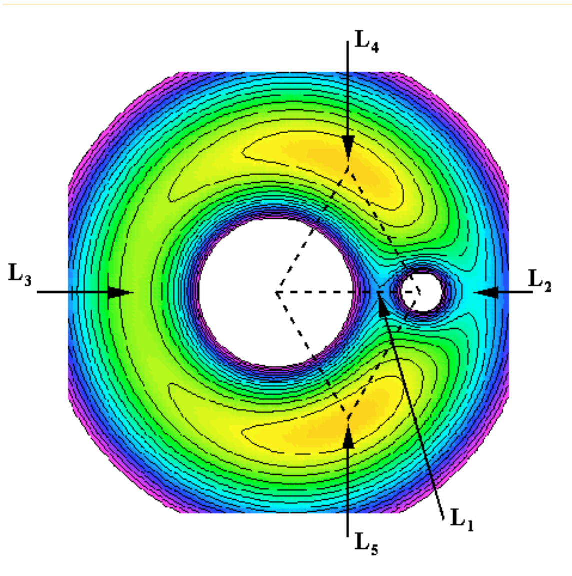
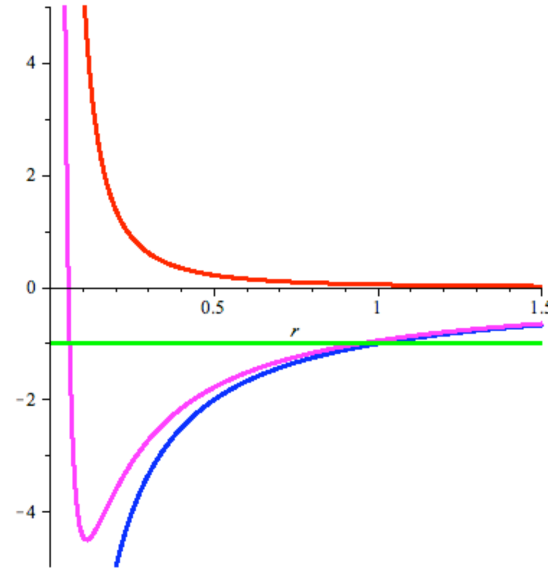


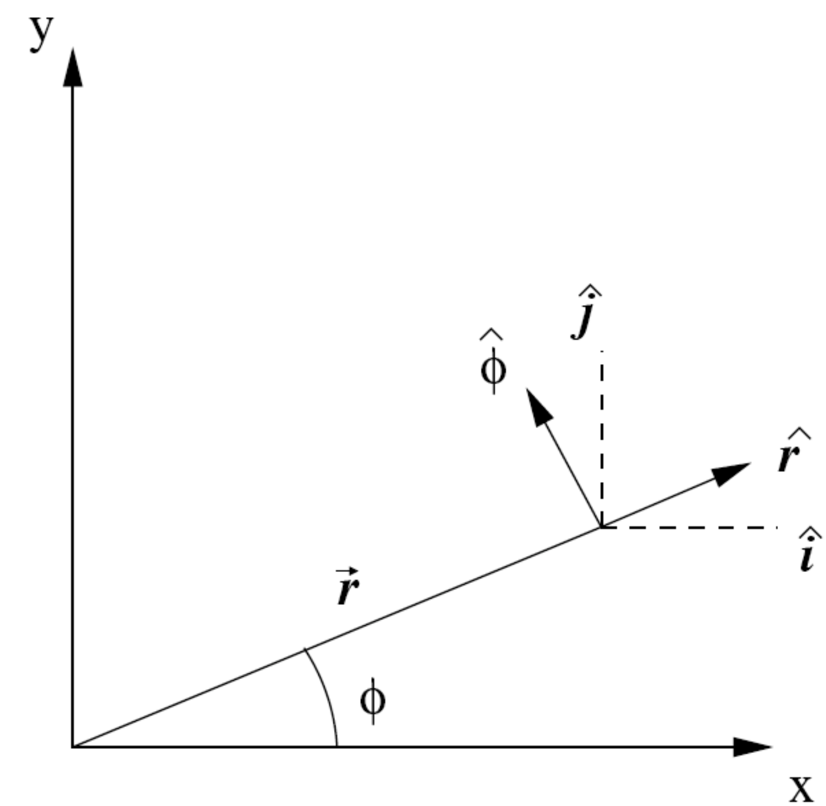


Effective Potentials

Reading: Course packet section 11

Taylor section 8.4

Velocity and acceleration in polar coordinates



$$\dot{\hat{r}} = \dot{\phi}\hat{\phi}$$

$$\dot{\hat{\phi}} = -\dot{\phi}\hat{r}$$

$$\frac{d\vec{r}}{dt} \equiv \dot{\vec{r}} = \vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

$$\frac{d^2\vec{r}}{dt^2} \equiv \ddot{\vec{r}} = \vec{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

Energy approach for fictitious reduced mass:

$$f(r)\hat{r} = -\nabla U(r); E = KE + U$$

$$E = \underbrace{\frac{1}{2}\mu\dot{\vec{r}}\bullet\dot{\vec{r}}}_{\text{KINETIC ENERGY}} + \underbrace{U(r)}_{\text{POTENTIAL ENERGY}} \quad ;\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

$$KE = \frac{1}{2}\mu\left(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\right)\bullet\left(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\right)$$

$$= \underbrace{\frac{1}{2}\mu\dot{r}^2}_{\text{RADIAL}} + \underbrace{\frac{1}{2}\mu r^2\dot{\phi}^2}_{ROTATIONAL}$$

Aside: Angular momentum for the reduced mass problem (again!)

$$\vec{\ell} = \vec{r} \times \vec{p} = \vec{r} \times \mu \vec{v} = \vec{r} \times \mu \dot{\vec{r}}$$

$$= r\hat{r} \times \mu \left(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \right)$$

$$= -\mu r^2 \dot{\phi}\hat{\theta} = \mu r^2 \dot{\phi}\hat{z} = \mu r^2 \dot{\phi}\hat{k}$$

$$\ell = \left|\vec{\ell}\right| = \mu r^2 \dot{\phi} = const$$

Notice that if r and ϕ-dot (aka omega) are known at one time, then el is known at that time and thus for all times. INITIAL CONDITIONS!

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

Energy approach

$$E = \underbrace{\frac{1}{2}\mu\dot{\vec{r}}\bullet\dot{\vec{r}}}_{\text{KINETIC ENERGY}} + \underbrace{U(r)}_{\text{POTENTIAL ENERGY}} \quad ;\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

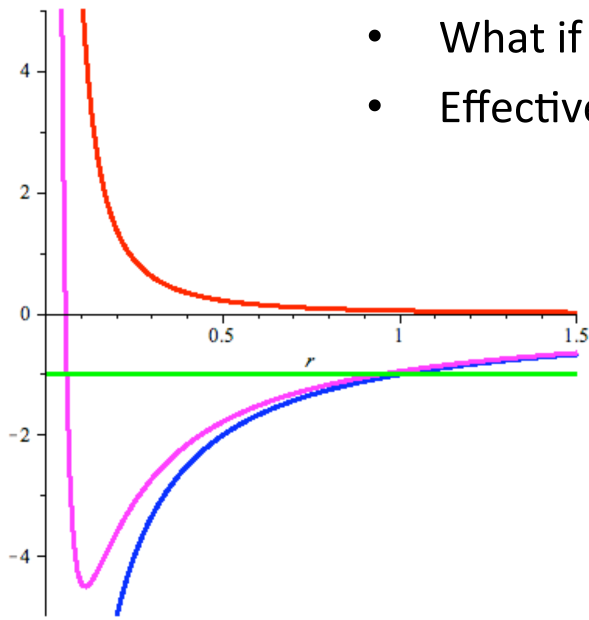
$$E = \underbrace{\frac{1}{2}\mu\dot{r}^2}_{\text{"LINEAR" KE}} + \underbrace{\frac{1}{2}\frac{\ell^2}{\mu r^2}}_{\text{ROTATIONAL KE}} \underbrace{+U(r)}_{\text{POT ENERGY}}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)}_{\text{EFFECTIVE POTENTIAL ENERGY}}$$

- This looks like $1/2*mv^2$ + effective pot energy

Energy approach: the effective potential

- Limits: Which part is big and which small as *r* changes?
- What is distance of closest approach?
- Are there energies where orbit is bound? Unbound?
- What if $E < U_{eff}(r)$?
- Effective potential Mathematica worksheet



$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)}_{\text{EFFECTIVE POTENTIAL ENERGY}}$$

Energy approach to find *r*(*t*); ϕ(*t*)

This is just as in PH421 where we treated the harmonic oscillator, we can solve for dr/dt as a function of *r*, and integrate to get *t*(*r*). And in PH421, we had 1-d so no angular momentum and used Harmonic Oscillator potential energy!

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

$$\dot{r} = \pm\sqrt{\frac{2E}{\mu} - \frac{2}{\mu}\left[U(r) + \frac{\ell^2}{2\mu r^2}\right]} = \pm\sqrt{\frac{2\left[E - V_{eff}(r)\right]}{\mu}}$$

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

Energy approach to find *r*(*t*); ϕ(*t*)

This is just as in PH421 where we treated the harmonic oscillator, we can solve for dr/dt as a function of *r*, and integrate to get *t*(*r*). Invert *t*(*r*) to get *r*(*t*). In PH421, we had 1-d so no angular momentum and we used the Harmonic Oscillator potential energy! In PH 421, we were especially interested in the period, so we integrated from one extreme of the motion to the other. You can do that here, if you like, but below we integrate from some arbitrary start to the point in question, *r*

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2E}{\mu} - \frac{2}{\mu}\left[U(r) + \frac{\ell^2}{2\mu r^2}\right]} \Rightarrow t = \int_0^r \frac{dr'}{\pm\sqrt{\frac{2E}{\mu} - \frac{2}{\mu}\left[U(r') + \frac{\ell^2}{2\mu r'^2}\right]}}$$

$$\dot{\phi} = \frac{\ell}{\mu r^2} \Rightarrow \phi(t) = \int_0^t \frac{\ell}{\mu r(t')^2}\,dt'$$

The orbit

- If we have $r(t)$ and $\phi(t)$, then we can call "t" a parameter and plot the orbit on a polar plot, with (r, ϕ) points for each value of t.
- Alternatively, we can solve one equation for t and plug into the other to get $r(\phi)$, say, and have the orbit directly.
- Even better, we can alter the original de's to get only r and ϕ, and solve that.
- (Recall the motion of a projectile in the earth's field for a simpler example: $x(t)$, $y(t)$ or $y(x)$)

Newton's Law for fictitious reduced mass

$$f(r)\hat{r} = \mu \frac{d^2\vec{r}}{dt^2}$$

$$= \mu\left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \mu\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

$$\frac{d}{dt}\left(\mu r^2\dot{\phi}\right) = 0 \Rightarrow \mu r^2\dot{\phi} \equiv \ell = \text{constant}$$

$$\ddot{r} = \frac{\ell^2}{\mu^2 r^3} + \frac{1}{\mu} f(r) \Rightarrow r(t) \qquad r(t) \Rightarrow \phi(t)$$