$$\vec{R}_{CM} \equiv \frac{m_1}{M}\vec{r}_1 + \frac{m_1}{M}\vec{r}_2 + \frac{m_1}{M}\vec{r}_3 + \ldots = \sum_n \frac{m_n}{M}\vec{r}_n \qquad \vec{L} = \vec{r}\times\vec{p}; \quad \vec{\tau} = \vec{r}\times\vec{F}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r) \qquad \vec{f}(r)\hat{r} = \mu\frac{d^2\vec{r}}{dt^2} = \mu\left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \mu\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

$$r(\phi) = \frac{\alpha}{1+\varepsilon\cos\phi}$$

$$\mathbf{r} = r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k} \qquad \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k} \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$$

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

$$\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)\Theta_\ell^m(\theta)\Phi_m(\phi) = R_{n\ell}(r)Y_\ell^m(\theta,\phi)$$

$$\psi(r,\theta,\phi,t) = \sum_{n\ell m} c_{n\ell m}R_{n\ell}(r)Y_\ell^m(\theta,\phi)e^{-iE_nt/\hbar}$$

$$L_x \doteq i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) \qquad L_z \doteq -i\hbar\frac{\partial}{\partial\phi}$$

$$L_y \doteq i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) \qquad \mathbf{L}^2 \doteq -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$H_{ring} = \frac{L_z^2}{2I} \qquad H_{sphere} = \frac{\mathbf{L}^2}{2I} \qquad \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

ℓ	m	$Y_\ell^m(\theta,\phi)$
0	0	$Y_0^0 = \sqrt{\frac{1}{4\pi}}$
1	0	$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$
	±1	$Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$
2	0	$Y_2^0 = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$
	±1	$Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi}$
	±2	$Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm i2\phi}$
3	0	$Y_3^0 = \sqrt{\frac{7}{16\pi}}\left(5\cos^3\theta - 3\cos\theta\right)$
	±1	$Y_3^{\pm1} = \mp\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta - 1\right)e^{\pm i\phi}$
	±2	$Y_3^{\pm2} = \sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta e^{\pm i2\phi}$
	±3	$Y_3^{\pm3} = \sqrt{\frac{35}{64\pi}}\sin^3\theta e^{\pm i3\phi}$

$$E_n = -\frac{1}{n^2}\frac{Z^2 e^4 m_e}{2\left(4\pi\varepsilon_0\hbar\right)^2} = -\frac{1}{n^2}\frac{1}{2}\alpha^2 m_e c^2; \quad n = 1,2,3...$$

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r} \qquad a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} \qquad \alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$$

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

$$\psi_{200}(r,\theta,\phi) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{2a_0}\right)^{3/2}\left[1-\frac{Zr}{2a_0}\right]e^{-Zr/2a_0}$$

$$\psi_{210}(r,\theta,\phi) = \frac{1}{2\sqrt{\pi}}\left(\frac{Z}{2a_0}\right)^{3/2}\frac{Zr}{a_0}e^{-Zr/2a_0}\cos\theta$$

$$\psi_{21,\pm1}(r,\theta,\phi) = \mp\frac{1}{2\sqrt{2\pi}}\left(\frac{Z}{2a_0}\right)^{3/2}\frac{Zr}{a_0}e^{-Zr/2a_0}\sin\theta e^{\pm i\phi}$$

$$\psi_{300}(r,\theta,\phi) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{3a_0}\right)^{3/2}\left[1-\frac{2Zr}{3a_0}+\frac{2}{27}\left(\frac{Zr}{a_0}\right)^2\right]e^{-Zr/3a_0}$$

$$\psi_{310}(r,\theta,\phi) = \frac{2\sqrt{2}}{3\sqrt{3\pi}}\left(\frac{Z}{3a_0}\right)^{3/2}\frac{Zr}{a_0}\left(1-\frac{Zr}{6a_0}\right)e^{-Zr/3a_0}\cos\theta$$

$$\psi_{31,\pm1}(r,\theta,\phi) = \mp\frac{2}{3\sqrt{3\pi}}\left(\frac{Z}{3a_0}\right)^{3/2}\frac{Zr}{a_0}\left(1-\frac{Zr}{6a_0}\right)e^{-Zr/3a_0}\sin\theta e^{\pm i\phi}$$

$$\psi_{320}(r,\theta,\phi) = \frac{1}{27\sqrt{2\pi}}\left(\frac{Z}{3a_0}\right)^{3/2}\left(\frac{Zr}{a_0}\right)^2 e^{-Zr/3a_0}\left(3\cos^2\theta-1\right)$$

$$\psi_{32,\pm1}(r,\theta,\phi) = \mp\frac{\sqrt{3}}{27\sqrt{\pi}}\left(\frac{Z}{3a_0}\right)^{3/2}\left(\frac{Zr}{a_0}\right)^2 e^{-Zr/3a_0}\sin\theta\cos\theta e^{\pm i\phi}$$

$$\psi_{32,\pm2}(r,\theta,\phi) = \frac{\sqrt{3}}{54\sqrt{\pi}}\left(\frac{Z}{3a_0}\right)^{3/2}\left(\frac{Zr}{a_0}\right)^2 e^{-Zr/3a_0}\sin^2\theta e^{\pm i2\phi}$$

$$\int xe^{-x}\,dx = -xe^{-x} - e^{-x} \qquad \int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} \qquad \int x^3 e^{-x}\,dx = -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x}$$

$$\int x^4 e^{-x}\,dx = -x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} - 24xe^{-x} - 24e^{-x}$$

$$\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$$