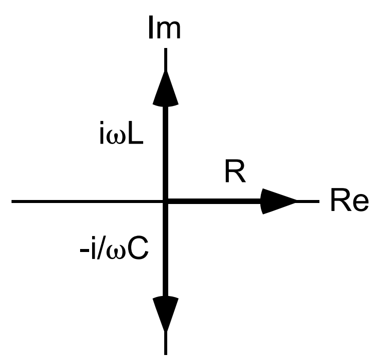
Phasor Analysis of Circuits

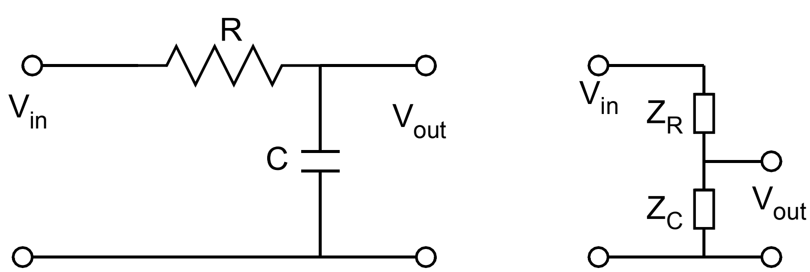
Concepts

Frequency-domain analysis of a circuit is useful in understanding how a single-frequency wave undergoes an amplitude change and phase shift upon passage through the circuit. The concept of impedance or reactance is central to frequency-domain analysis. For a resistor, the impedance is $Z_R(\omega) = R$, a real quantity independent of frequency. For capacitors and inductors, the impedances are $Z_C(\omega) = -i/\omega C$ and $Z_L(\omega) = i\omega L$. In the complex plane these impedances are represented as the phasors shown below.



These phasors are useful because the voltage across each circuit element is related to the current through the equation $V = I\,Z$. For a series circuit where the same current flows through each element, the voltages across each element are proportional to the impedance across that element.

Phasor Analysis of the *RC* Circuit



The behavior of this *RC* circuit can be analyzed by treating it as the voltage divider shown at right. The output voltage is then

$$\frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_C + Z_R} = \frac{-i/\omega C}{-i/\omega C + R}.$$

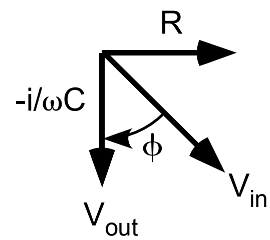
The amplitude is then

$$\left|\frac{V_{out}}{V_{in}}\right| = \left|\frac{-i}{-i + \omega RC}\right| = \left|\frac{1}{1 + i\omega/\omega_c}\right| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}},$$

where we have defined the corner, or 3dB, frequency as

$$\omega_c = \frac{1}{RC}.$$

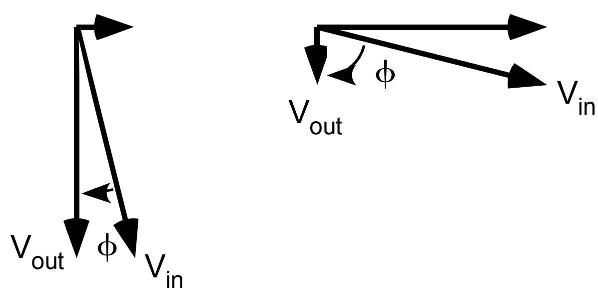
The phasor picture is useful to determine the phase shift and also to verify low and high frequency behavior. The input voltage is across both the resistor and the capacitor, so it is equal to the vector sum of the resistor and capacitor voltages, while the output voltage is only the voltage across capacitor. These voltages are represented by the phasors shown below:



The phase shift of the output voltage compared to the input voltage is represented by the angle ϕ shown. In the case shown, the output lags behind the input, so the phase shift is negative ($-90^\circ < \phi < 0$). The reason that the output is behind the input is that the phasor picture assumes that the nominal time dependence is $e^{i\omega t}$, which is represented by a phasor moving in the counterclockwise direction (hence the clockwise ϕ shown is negative). To calculate the phase shift, we use trig to get

$$\tan(-\phi) = \frac{R}{1/\omega C} = \omega RC = \frac{\omega}{\omega_c}$$
$$\Rightarrow \phi = -\tan^{-1}\left(\frac{\omega}{\omega_c}\right).$$

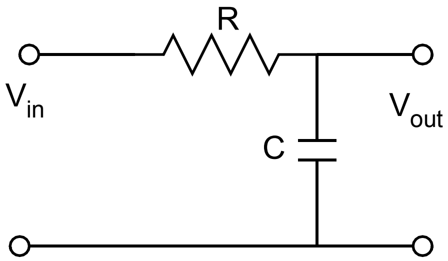
To see the low and high frequency behavior, consider the phasor diagrams where $\omega \ll \omega_c$ (left) and $\omega \gg \omega_c$ (right), meaning that the capacitive reactance dominates the resistive reactance and vice versa:

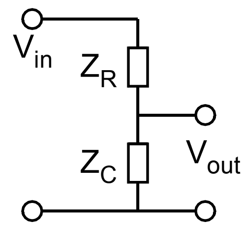


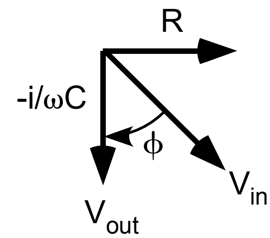
The diagrams demonstrate that at $\omega \ll \omega_c$: $V_{out} \simeq V_{in}$ and $\phi \to 0$ and at $\omega \gg \omega_c$: $V_{out} \to 0$ and $\phi \to -90^\circ$.

This case and the other filter cases are summarized on the following pages.

Low-Pass RC circuit $\qquad \omega_c = \dfrac{1}{RC}$







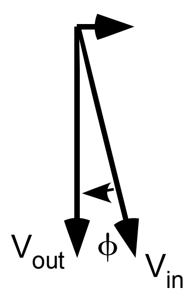
$$\frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_C + Z_R} = \frac{-i/\omega C}{-i/\omega C + R}$$

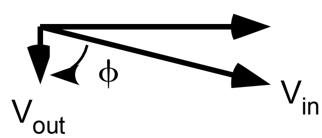
$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1+\left(\omega/\omega_c\right)^2}}$$

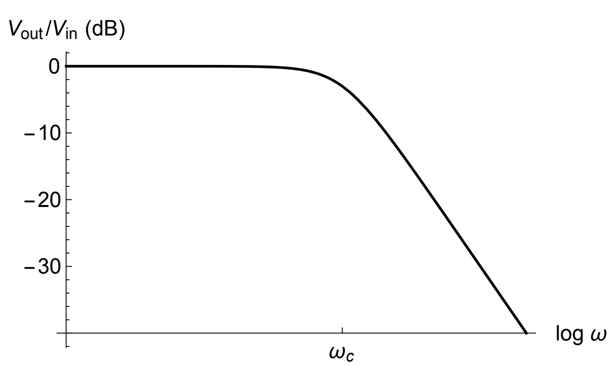
$$\tan\left(-\phi\right) = \frac{R}{1/\omega C} = \omega RC = \frac{\omega}{\omega_c}$$

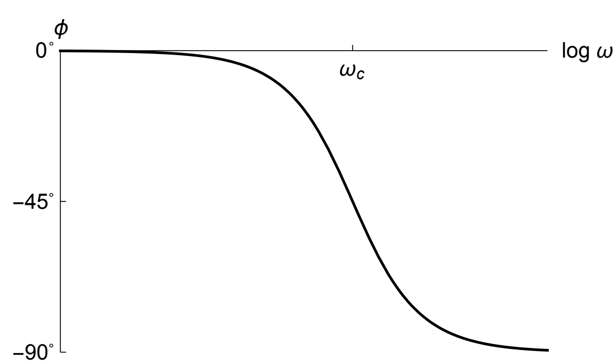
$$\Rightarrow \phi = -\tan^{-1}\left(\frac{\omega}{\omega_c}\right)$$

$\omega \ll \omega_c$	$\omega \gg \omega_c$
$V_{out} \simeq V_{in}$	$V_{out} \to 0$
$\phi \to 0^\circ$	$\phi \to -90^\circ$

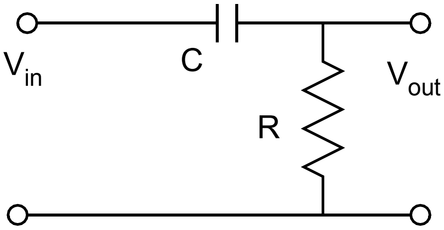


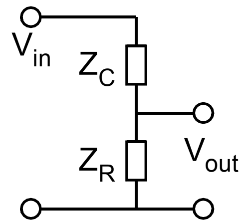


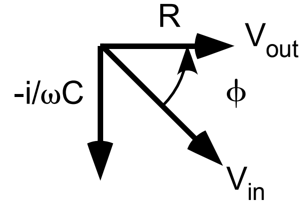




High-Pass CR circuit $\qquad \omega_c = \dfrac{1}{RC}$







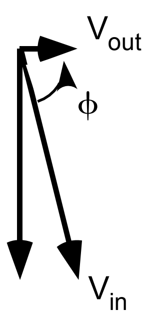
$$\frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_C + Z_R} = \frac{R}{-i/\omega C + R}$$

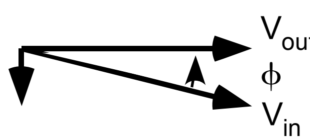
$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{(\omega/\omega_c)}{\sqrt{1+(\omega/\omega_c)^2}}$$

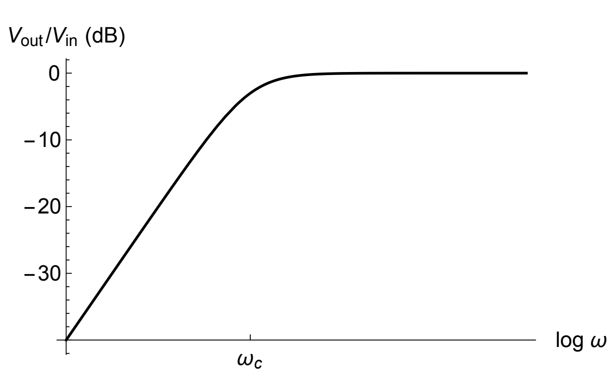
$$\tan\phi = \frac{1/\omega C}{R} = \frac{1}{\omega RC} = \frac{\omega_c}{\omega}$$

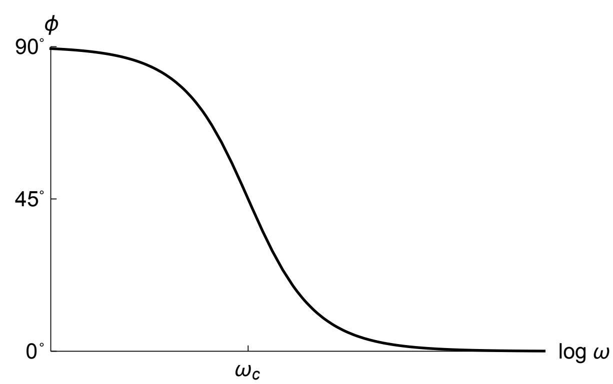
$$\Rightarrow \phi = \tan^{-1}\left(\frac{\omega_c}{\omega}\right)$$

$\omega \ll \omega_c$	$\omega \gg \omega_c$
$V_{out} \to 0$	$V_{out} \simeq V_{in}$
$\phi \to 90^\circ$	$\phi \to 0^\circ$

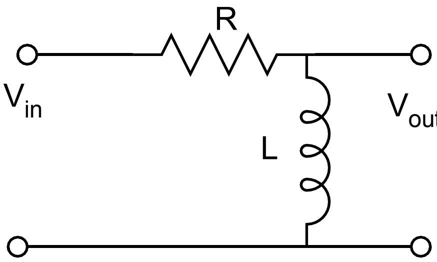


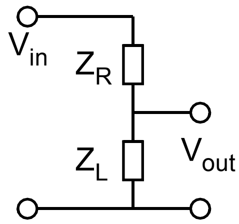


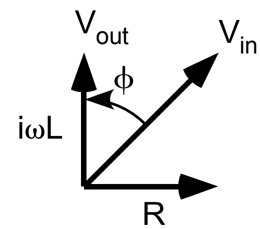




High-Pass RL circuit $\qquad \omega_c = \dfrac{R}{L}$







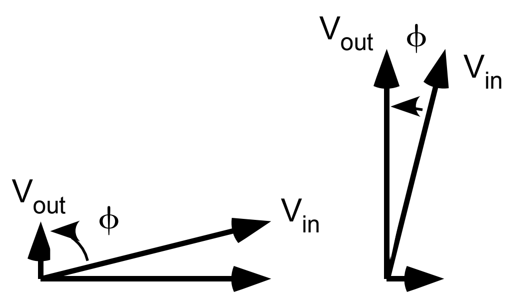
$$\frac{V_{out}}{V_{in}} = \frac{Z_L}{Z_L + Z_R} = \frac{i\omega L}{i\omega L + R}$$

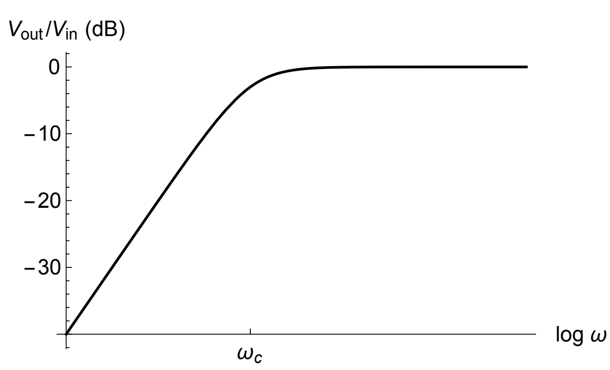
$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{(\omega/\omega_c)}{\sqrt{1+(\omega/\omega_c)^2}}$$

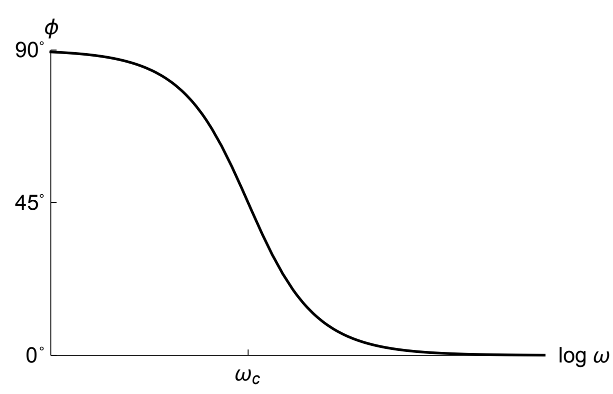
$$\tan\phi = \frac{R}{\omega L} = \frac{R/L}{\omega} = \frac{\omega_c}{\omega}$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{\omega_c}{\omega}\right)$$

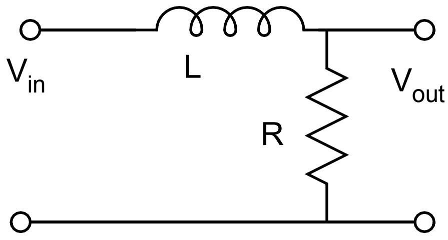
$\omega \ll \omega_c$	$\omega \gg \omega_c$
$V_{out} \to 0$	$V_{out} \simeq V_{in}$
$\phi \to 90^\circ$	$\phi \to 0^\circ$

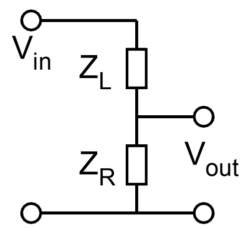


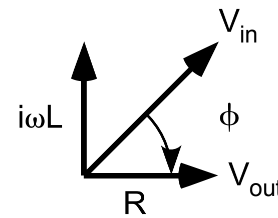




Low-Pass LR circuit $\omega_c = \frac{R}{L}$







$$\frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_L + Z_R} = \frac{R}{i\omega L + R}$$

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1+\left(\omega/\omega_c\right)^2}}$$

$$\tan(-\phi) = \frac{\omega L}{R} = \frac{\omega}{R/L} = \frac{\omega}{\omega_c}$$

$$\Rightarrow \phi = -\tan^{-1}\left(\frac{\omega}{\omega_c}\right)$$

$\omega \ll \omega_c$	$\omega \gg \omega_c$
$V_{out} \simeq V_{in}$	$V_{out} \to 0$
$\phi \to 0$	$\phi \to -90^\circ$