NORMAL SUBGROUPS OF THE FREE GROUP

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ABSTRACT. We present a method of producing normal subgroups of the free group on r generators not containing a given word. Special attention is given to the normal subgroups of the free group on two generators. In considering the exponent sum of the generators in a given word, we produce such subgroups and clearly state for what class of words these subgroups are not easily obtainable. We also provide a general condition for skipping in $p \times q$ covers with a backbone, providing conditions in which skipping covers may be generalized to higher folds. Furthermore, we analyze certain infinite families of covers and show their relevance in enumerating covers of a given fold.

1. INTRODUCTION

Given the free group on two generators a and b (henceforth denoted by $F(2)$), it has been shown by algebraic means that for any word $g \in F(2)$ there exists a normal subgroup H of finite index such that $g \notin H$ [1]. However, no method exists for producing such a subgroup. To this end, we will make extensive use of the well-known correspondence between normal subgroups of the fundamental group of a graph and its regular covering spaces. We consider normal subgroups of the free group on r generators corresponding to regular coverings of the wedge of r circles, focusing primarily on the free group on two generators and the wedge of 2 circles and generalizing whenever possible.

In section 2, we discuss a method for producing a regular cover in which a given word does not lift to a loop, thereby bounding the minimal index of a normal subgroup not containg the word. Further, we show that for certain classes of words, the normal subgroup produced is of minimal index. The initial method provides desirable covers for several, but not all classes of words. In section 2.1, we discuss a class of words in which both generators' exponent sum is zero, and provide normal subgroups not containing these words. What remains are words which cannot be handled by covers corresponding to elementary groups. We then consider covers corresponding to more complex groups (such as A_5) in section 2.2, and also which words do not lift to loops in such covers. Lastly, section 3 expands on observations about covers described by Nieveen and Smith [2], and section 4 generalizes certain types of covers into infinite families.

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2. Normal subgroups of the free group not containing a given word

We begin by defining some characteristics of elements of the free group. Throughout the paper, free-reduced elements of $F(2)$ shall be referred to as words. Define the *a-length* of a given word to be the sum of the powers of a, with *b-length* defined in a similar fashion. Denote the a-length by p and the b-length by q.

Given a word on two generators, there exists a regular covering of the figure-eight space in which the word does not lift to a loop. The following proof demonstrates a process for constructing such a cover when p and q are not both zero, thereby producing an upper bound for the minimal index of the corresponding normal subgroup of F.

Proposition 2.1. *Let $g \in F(2)$ with p and q not both zero, n be the smallest non-divisor of p, and m be the smallest non-divisor of q. There exists $H \triangleleft F$ of index* $\min\{m,n\}$ *such that $g \notin H$.*

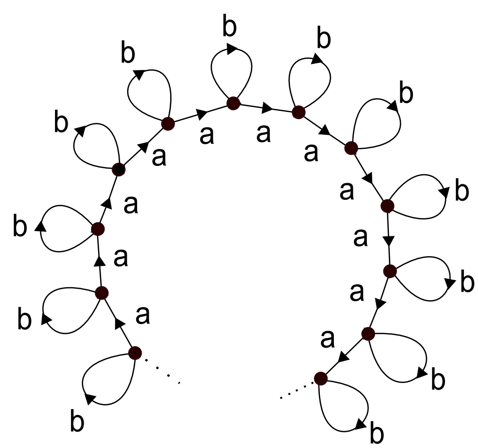

FIGURE 1. Simple r-gon

Consider the regular r-fold cover of the figure eight in Figure 1. We will refer to a regular cover of this type as a *simple r-gon*. Note that this cover, having one generator of order r (forming the r-gon), and the other of order 1 (forming the loops at each vertex), corresponds to the group presentation $\langle a,b|\ a^r,b\rangle \cong \mathbb{Z}_r$. The symmetry of the cover allows any vertex to be mapped to any other vertex, implying regularity (see Note 5.4.)

Proof. Without loss of generality, assume $n \leq m$. Consider the simple n-gon corresponding to the group presentation $\langle a,b|\ a^n,b\rangle$. Fix an arbitrary initial vertex. Given $g \in F(2)$ and traversing the corresponding path in the cover, each b-edge will return you to the same vertex. Thus b-edges do not affect where the path terminates. Therefore, in considering which words will form a loop in the cover, we need not consider b-edges. Hence we will have an a-loop in the simple n-gon if and only if $p \equiv 0 \mod n$. Since n is the smallest non-divisor of p, we have that $p \not\equiv 0 \mod n$ and the path will not terminate at the initial vertex. This implies g is not contained in the normal subgroup corresponding to the symmetries of the simple n-gon. □

Now we show that in certain cases, there exists no normal subgroup with a smaller index not containing g.

Suppose first that p or q is odd. Then $n = 2$ and there exists an index 2 normal subgroup that does not contain g by the construction above. There is no nontrivial subgroup with smaller index and so the construction above yields the minimal index subgroup not containing g.

Now suppose that $n = 3$; then p and q are both even. As we have shown, there exists a normal subgroup of index 3 not containing g. What must be shown is that no normal subgroup of index 2 exists that does not contain g. There are only three regular 2-fold covers [2, p. 107], as in Figure 2 below. Thus there are only three normal subgroups of $F(2)$ with index 2. Two of these covers are simple 2-gons, and g will lift to a loop in both of them, as $p \equiv q \equiv 0 \mod 2$. Thus the only remaining cover is on the right in the figure below.

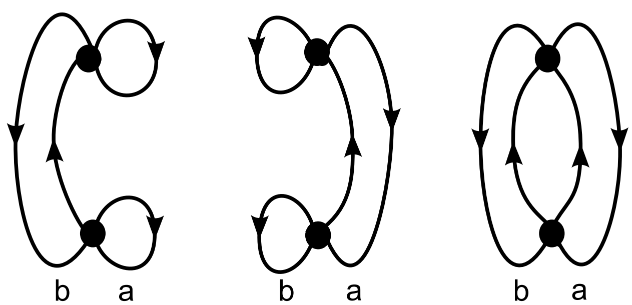

FIGURE 2. Two-fold cover with $a^2 = b^2 = 1$.

Since $n = 3$, both p and q are even, and thus $p \equiv q \equiv p + q \equiv 0 \mod 2$. As above, g is a loop in this cover.

Thus the smallest index normal subgroup that does not contain g must be 3.

We now address the entire class of covers whose corresponding symmetry groups are abelian.

Consider g with p and q both non-zero multiples of 6. We have shown that there is a cover of size n such that the corresponding normal subgroup does not contain this word. It remains to be shown that there is no r-fold cover with $r < n$, where g does not lift to a loop in the cover. We now address the subclass of these covers whose corresponding symmetry groups are abelian.

Theorem 2.2. *Given $g \in F(2)$, the symmetry group of the simple n-gon is of minimal index among all normal subgroups corresponding to abelian symmetry groups in which g does not lift to a loop.*

Proof. Let $r < n$. Consider an r-fold regular cover whose symmetries correspond to an abelian normal subgroup. Suppose $o(a) = m_1$ and $o(b) = m_2$, and let g have a-length p and b length q. We have chosen n to be the least non-divisor of p and q, so $r|p$ and $r|q$. Further, $m_1|r$ and $m_2|r$ [2, pp. 114-115], so there exist $k_1, k_2 \in \mathbb{Z}$ with $p = k_1 m_1$ and $q = k_2 m_2$. Note that $a^{k_1 m_1} b^{k_2 m_2}$ must form a loop, since $a^{k_1 m_1} b^{k_2 m_2} = (a^{m_1})^{k_1} (b^{m_2})^{k_2} = 1^{k_1} 1^{k_2} = 1$. Because the subgroup corresponding to this regular cover is abelian, any word with a-length $k_1 m_1$ and b-length $k_2 m_2$ must form a loop, and thus g will lift to a loop in this cover. Therefore there exists no regular r-fold cover with $r < n$ corresponding to an abelian group which does not contain g. □

Corollary 2.3. *The symmetry group of the simple n-gon, where* $n = \min\{m, n\}$ *is as defined in the proof of Proposition 2.1, corresponds to the minimal index normal subgroup for all words in which p and q are not both multiples of 60.*

By inspection, all symmetry groups corresponding to n-fold covers with $n < 6$ are abelian. Thus by the theorem, if the procedure above yields a simple n-gon cover with $n < 6$, it is the minimum fold cover for the given word.

In order for the simple n-gon to not necessarily be the minimum fold cover, n must be divisible by 2, 3, 4, and 5. Thus for all words with n not a multiple of 60, the simple n-gon is the minimal fold desirable cover.

2.1. **Words contained in the commutator subgroup of F(2).** The remaining words are those with both p and q are zero. This case corresponds to words contained in $F(2)'$, the commutator subgroup of $F(2)$. We observe that there are subclasses of words with $p = 0$, $q = 0$ (referred to as *0-0 words*) for which a bound on the minimal index normal subgroup can be easily obtained.

Definition 2.4. Nested n-gons *are* $2n$*-fold covers which correspond to the symmetries of the dihedral group as shown in Figure 3 below. These covers are a subclass of* double n-gon *covers, which will be addressed in Section 4.2*

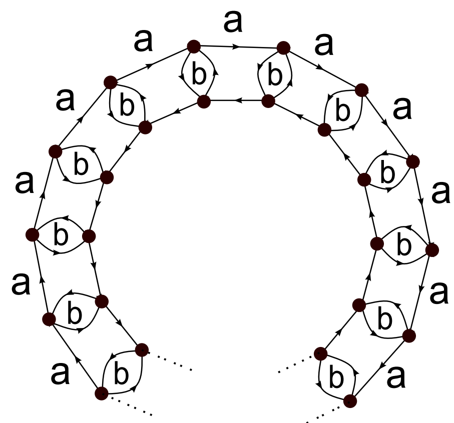

FIGURE 3. Nested n-gon

Definition 2.5. *Let $g = a^{i_1}b^{j_1}a^{i_2}b^{j_2}\ldots a^{i_k}b^{j_k}$ be a generic word in $F(2)$. Define a^*-*reduction *by identifying b^{2m} with the trivial word, followed by free reduction as usual, repeating the process until the word is completely reduced. Denote the a^*-reduced form of g by g_a^*. Define b^*-*reduction *and g_b^* similarly.*

Note that a word can be a^*-reduced without changing its properties relative to the nested n-gon (i.e. the word's terminal vertex from a given initial vertex), since the b-edges form 2-cycles, making the action of b^2 identical to the action of the trivial word. Thus the action of b^{2m} is equivalent to the trivial word acting m times.

Lemma 2.6. *Let $g \in F(2)$ and let $g_a^* = a^{i_1^*}b^{j_1^*}a^{i_2^*}b^{j_2^*}$ be its a^*-reduced form. Then g does not lift to a loop in a nested n-gon if $\sum_{m=1}^{k}(-1)^{m+1}i_m^* \not\equiv 0 \mod n$.*

Proof. Consider the nested n-gons above. We will refer to the "inside" and "outside" of this cover throughout this proof, keeping in mind that they are equivalent by regularity. Let the initial vertex, v_0, lie on the outside n-gon. Without loss of generality, let the outside n-gon be oriented clockwise. Let clockwise motion along an a-edge be represented by a value of $+1$. Moving along an a-edge in the inner n-gon then represents a value of -1, as this is equivalent to moving counterclockwise along an a-edge on the outer n-gon (up to corresponding vertices between the two n-gons). Note that since $q = 0$, g will touch an even number of b-edges, and thus the path necessarily terminates on the outer n-gon. We can sum (modulo n) the values obtained from the a-edges in order to evaluate how many vertices away from v_0 the terminal vertex lies. Supposing that $\sum_{m=1}^{k}(-1)^{m+1}i_m^* \not\equiv 0 \mod n$, the path cannot terminate at v_0, and hence g_a^* does not form a loop in the nested n-gon. By the definition of a^*-reduction, g therefore also does not lift to a loop in the cover. □

Using the preceding lemma, we may systematically check if a given 0-0 word will lift to a non-loop in some nested n-gon. Given g with the 0-0 property, we first perform a^*-reduction. We then check to see if $\sum_{m=1}^{k}(-1)^{m+1}i_m^* \neq 0$ for g_a^*. If it is non-zero, we choose the smallest n such that $\sum_{m=1}^{k}(-1)^{m+1}i_m^* \not\equiv 0 \mod n$. If the sum is equal to zero, we then return to the original word g and perform b^*-reduction, again checking if the sum is non-zero and applying the same rule for choosing n. If the sum is equal to zero for both g_a^* and g_b^*, then the word will lift to a loop for all nested n-gons. However, if an n value is obtained, we have found (by correspondence) a normal subgroup N, with $[F(2):N] = 2n$, such that $g \notin N$.

2.2. **The alternating group on 5 symbols.** Thus far we have considered covers whose symmetries correspond to groups having the metabelian property (i.e. cyclic, generalized quaternion, and dihedral groups). It can be shown that words in $F(2)''$, as well as some words in $F(2)'$, will lift to loops in covers whose symmetries correspond to metabelian groups. For this reason we consider *simple groups* on two generators in order to produce covers for which these words will lift to non-loops.

Consider the group of symmetries of the dodecahedron given by A_5, the alternating group on five symbols, with group presentation $\langle a,b \mid a^2, b^3, (ab)^5\rangle$. The relations for this group of symmetries are produced by two covers (one corresponding to this presentation with a and b interchanged). One such cover is represented below.

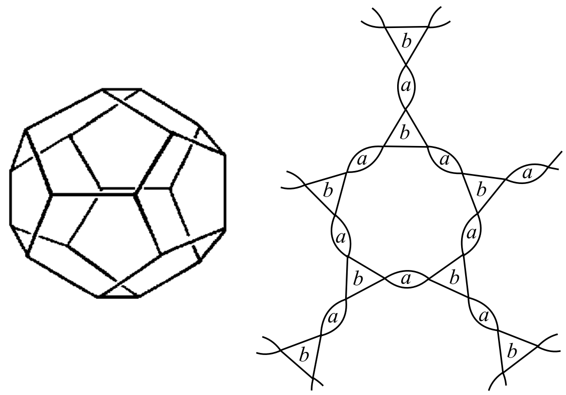

FIGURE 4. (left) A dodecahedron. (right) Cover whose symmetries correspond to the alternating group on five symbols.

Since A_5 is a simple group, it is not metabelian, and thus most 0-0 words will lift to non-loops. Given a word $g \in F(2)'$, we consider the cover corresponding to the above presentation: $\langle a, b | a^2, b^3, (ab)^5 \rangle$. Using the relations, we simplify g, followed by free reduction, until g is completely reduced. If what remains is non-trivial, then the word lifts to a non-loop in the corresponding cover. If g is found to be trivial, consider the cover with the a-edges and b-edges switched, corresponding to $\langle a, b | a^3, b^2, (ba)^5 \rangle$. There are words in $F(2)'$, such as those of the form $g = [r_i^{6m}, r_j^{6n}]$ (where r_k corresponds to a relation in either presentation of A_5) which are trivial in both presentations. It is useful to assert exactly what subset of words are trivial in any presentation of A_5.

Conjecture 2.7. *A word $g \in F(2)$ is trivial in any presentation of A_5 if $g \in S = \langle\langle a^2, b^3, (ab)^5 \rangle\rangle \cap \langle\langle b^2, a^3, (ba)^5 \rangle\rangle$ where $\langle\langle\ \rangle\rangle$ is the normal closure in $F(2)$.*

This argument can be strengthened by considering the binary icosahedral group (with order 120), given by $\langle a, b |\ a^2 = b^3 = (ab)^5 \rangle \cong SL(2,5)$ (where the relations are now equivalent to eachother, but not necessarily the identity). Consider the group $\bar{S}$ given by
$\langle\langle a^2\bar{b}^3, \bar{b}^3 a^2, \bar{b}^3 (ab)^5, (ab)^5 \bar{b}^3, \bar{a}^2 (ab)^5, (ab)^5 \bar{a}^2 \rangle\rangle \cap \langle\langle b^2 \bar{a}^3, \bar{a}^3 b^2, \bar{a}^3 (ba)^5, (ba)^5 \bar{a}^3, \bar{b}^2 (ba)^5, (ba)^5 \bar{b}^2 \rangle\rangle$. (This group is analogous to S as it is the normal closure of the relations derived from $SL(2,5)$.) We conjecture that $\bar{S} \subseteq S$, however the converse is not true (considering, for example, $a^2 \in S$ but $a^2 \notin \bar{S}$). Thus, more words lift to non-loops in the cover whose symmetries correspond to $F(2)/\bar{S} \cong SL(2,5)$ than in A_5.

However, for our purposes in bounding the minimal index normal subgroup not containing a given word, we need only consider elements of $F(2)'$ (since for any word with non-zero a- or b-length such a normal subgroup can be easily produced). Therefore the words which lift trivially in every cover examined thus far must be in $\bar{S} \cap [F(2), F(2)]$.

3. GENERALIZED SKIPPING IN COVERS WITH A BACKBONE

For this discussion we limit ourselves to $p \times q$ covers where $p \leq q$ and consider covers which only skip (and do not twist) [2, p.116].

Definition 3.1. *A connected, regular $p \times q$ cover has a* backbone *if $o(a) = p$, $o(b) = q$ and there is a b-cycle that visits every a-cycle.*

Note that by regularity, every b-cycle visits every a-cycle.

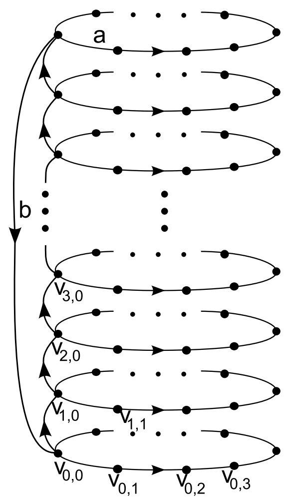

FIGURE 5. Cover with a backbone

Lemma 3.2. *Given a regular, connected $p \times q$ cover with a backbone, there may be skipping by n (as defined in* [2]*) in the cover if and only if $n^p \equiv 1 \mod q$.*

Proof. Consider a regular $p \times q$ cover. In order for the cover to have a backbone, each a-cycle must be connected to all other a-cycles by a sequence of b-edges. We may arrange the a-cycles so that there exists a b-cycle which has skipping by one. Without loss of generality let this cycle go through vertex $v_{(0,0)}$ (where (B,A) represents the position of the vertex in terms of the number of b- and a-edges away from $v_{(0,0)}$, respectively). Thus the b-edge from $v_{(0,0)}$ terminates at $v_{(1,0)}$. If skipping by n occurs, then vertex $v_{(0,1)}$ is joined to vertex $v_{(n,1)}$ by a b-edge. Therefore, the word $ab\bar{a}\bar{b}^n$ forms a loop in the cover. By regularity, this word must form a loop from all vertices and thus a b-edge from vertex $v_{(0,2)}$ must terminate at vertex $v_{(n^2,2)}$. In general, a b-edge originating from vertex $v_{(j,k)}$ must terminate at vertex $v_{(j+n^k \mod q,k)}$. Because the a-edges form a cycle of length p, $v_{(j,k)} = v_{(j,k+p)}$. In particular, $v_{(0,0)} = v_{(0,p)}$. By the above argument, the b-edge originating at vertex $v_{(0,p)}$ terminates at vertex $v_{(n^p \mod q,0)}$. However, as stated above, the b-edge from this vertex terminates at $v_{(1,0)}$. Thus we must have $n^p \equiv 1 \mod q$. □

This lemma provides us with one condition for which $p \times q$ covers may have skipping (which is, in fact, a necessary condition for $p \times q$ covers *with a backbone*). Also, certain small index covers with skipping can be seen to generalize to skipping covers of higher order. This condition can be used to make connections between covers of different varieties. We provide some of these details below.

3.1. **Conditions which allow skipping in covers with a backbone.**

(1) Let p and q be primes such that $p|(q-1)$. There exists a cover that skips.

(2) Let p be a prime and let $q = p^z$ for some integer $z > 1$. Then there is $p \times p^z$ cover with skipping.

(3) Let p and q be positive integers such that some $p \times q$ cover has skipping. Then skipping is allowed in a $kp \times q$ cover for all $k \in \mathbb{N}$.

(4) If skipping by n is allowed in a $p \times q$ cover, then $\forall\, k \in \mathbb{N}$ with $(k,q) = 1$, $\exists\, j \in \mathbb{N}$ such that skipping by $n + jq$ is allowed in a $p \times kq$ cover.

(5) In a $p \times mp$ cover for p prime and $m \in \mathbb{N}$, if skipping by n is allowed, then $n \equiv 1 \mod p$.

Proof. We now provide the proofs for each of the conditions above.

(1) This result was previously shown. [2, pp.118-9]

(2) Let $n = p^{z-1} + 1$. By the above lemma, to show that the $p \times p^z$ cover may have skipping by n, we must show that $n^p \equiv 1 \mod p^z$, or equivalently, $(p^{z-1} + 1)^p \equiv 1 \mod p^z$. We may simplify this by the binomial theorem:

$$(p^{z-1}+1)^p = \sum_{i=0}^{p} \binom{p}{i} p^{(z-1)i} = p^{(z-1)p} + \binom{p}{1} p^{(z-1)(p-1)} \cdots \binom{p}{p-1} p^{(z-1)} + 1$$

Note that, excluding the last two terms if this sum, each term has a power of p greater than z, and is thus divisible by p^z. Given that $\binom{p}{p-1} = p$, the next to last term is $\binom{p}{p-1} p^{(z-1)} = pp^{(z-1)} = p^z$. Thus $p^z|(p^{z-1}+1)^p - 1$ and $(p^{z-1}+1)^p \equiv 1 \mod p^z$, and so skipping by $(p^{z-1}+1)$ may occur in a $p \times p^z$ cover.

(3) Because there is skipping in the $p \times q$ cover, by Lemma 3.2 there exists an n such that $n^p \equiv 1 \mod q$. Thus $n^{pk} \equiv (n^p)^k \equiv (1)^k \equiv 1 \mod q$.

(4) This is equivalent to the following statement:
If, for a given p and q, there exists n such that $n^p \equiv 1 \mod q$, then for a given k with $(k,q) = 1$, there exists a j such that $(n + jq)^p \equiv 1 \mod kq$.

Consider the set $N = \{n, n+q, n+2q, \ldots, n+(k-1)q\}$. For all $x, y \in N, x \not\equiv y \mod k$ (since k and q are relatively prime). Note that $|N| = k$, so there exists $x \in N$ such that $x \equiv 1 \mod k$. Let $x = n + jq$. Then $(n+jq)^p \equiv 1 \mod k$.
$n + jq \equiv n \mod q$ and $n^p \equiv 1 \mod q \implies (n+jq)^p \equiv 1 \mod q$
Since $k|(n+qj)^p - 1$ and $q|(n+qj)^p - 1$, and $(k,q) = 1$, we know that $kq|(n+qj)^p - 1$ (Appendix (1)), as desired.

Note

We can find j explicitly. We have $n + jq \equiv 1 \mod k \iff jq \equiv 1 - n \mod k$. Since $(k,q) = 1$, there exists $a, b \in \mathbb{Z}$ such that $ak + bq = 1$. Looking at this equation modulo k, this is equivalent to finding $b \in \mathbb{N}$ such that $bq \equiv 1 \mod k$. Hence, $b \equiv q^{-1} \mod k$. So we can multiply the above equation by q^{-1} to get $j \equiv (1-n)q^{-1} \mod k$.

(5) Let $(mp)|n^p - 1$, and so $p|n^p - 1$. Assume for contradiction that $p \nmid n-1$.
Then $p \nmid (n-1)^p$
$\implies p \nmid \sum_{i=0}^{p} \binom{p}{i} n^i (-1)^{p-i}$
$\implies p \nmid -1 + \sum_{i=1}^{p-1} \binom{p}{i} n^i (-1)^{p-i} + n^p$
$\implies p \nmid -1 + n^p$ since all terms in the above summation are multiples of p.
But we know that $p| n^p - 1$, so this is a contradiction. Hence $p| n-1$.

□

4. Infinite families of covers

4.1. **Planar skipping.** Arrange n vertices into an n-sided polygon, and without loss of generality, adjoin each vertex to the successive vertex in the counterclockwise direction by an oriented a-edge. Now choose a vertex arbitrarily and label it v_0.

Definition 4.1. *Connect v_0 to one of the other $(n-1)$ vertices with a directed b-edge. This yields the relation $b = a^k$ for some k. We say that the cover has* planar skipping *(PS) with skipping value k.*

Given *PS* by k, regularity forces all b-edges to have the same skipping value at each vertex. For some values of k, this will create disjoint cycles of order less than n. Also, it is possible to skip in such a fashion that each b-cycle will have order n as well. We examine this skipping property in the following lemma.

Lemma 4.2. *Given an n-gon comprised of a-edges (as described above), with b-edges skipping by k, if $gcd(n,k) = 1$, then $o(b) = n$.*

Proof. Construct an n-gon with directed a-edges such that $a^n = 1$. Arbitrarily label a vertex v_0 and construct a b-edge that skips by k to vertex v_k such that $b = a^k$. Now complete a b-cycle by constructing a directed b-edge from v_k to v_{2k}, and continue in this fashion until v_0 is reached. Now suppose $gcd(n,k) = 1$. Thus we know that $o(b) = m$, where m is the smallest positive integer such that $mk \equiv 0 \mod n$. It is clear that $nk \equiv 0 \mod n$. Now suppose $pk \equiv 0 \mod n$ for some $p < n$. Thus $r| pk$ and since $gcd(n,k) = 1$ we know that $n| p$. However, $p < n$ and this yields a contradiction. Thus b must also have order n. □

Now using the previous lemma we will show that for *PS* covers with skipping number k, where n and k are relatively prime, the set of a-edges is interchangeable with the set of b-edges.

Lemma 4.3. *Given a n-gon with PS by k, where n and k are relatively prime, interchanging all a-edges with b-edges yield an equivalent cover.*

Proof. Note that in the above construction for an n-gon with *PS*, there is an a-edge from v_j to v_{j+1}, denoted by (v_j, v_{j+1}). Furthermore, a b-edge occurs between v_j and $v_{(j+1)k \mod n}$. Also note that by the above lemma, n is the order of both a and b. Now define a function $f : V \to V$, where V is the set of vertices, by $f(v_i) = v_{ik \mod n}$ for a given vertex v_i. This map is a bijection, and maintaining the stated rules for a- and b-edges, the permutation of vertices has produced an equivalent cover. Note that by this equivalence the geometric roles of a- and b-edges have been exactly interchanged. □

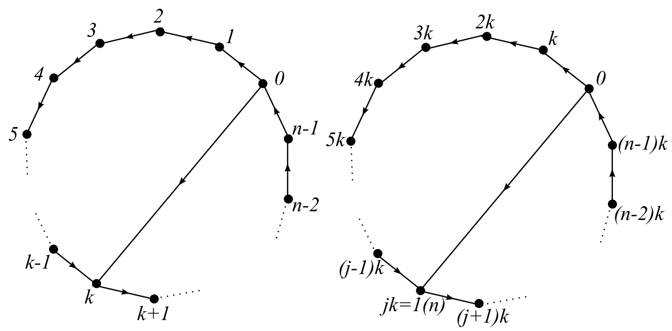

FIGURE 6. Planar skipping equivalence

The previous lemmas have provided a way to enumerate the *PS* covers for a given *n* value.

Theorem 4.4. *There are exactly* $2n-\Phi(n)$ *distinct n-gons with PS, where* Φ *is the Euler Function.*

Proof. Construct an n-gon consisting of a-edges, and without loss of generality orient them counterclockwise. Now, to enumerate the number of possible covers with *PS*, we consider the skipping values $k = 0,1,2,...,n-1$ for the placement of b-edges. Each value of k will produce a distinct cover. Thus we have n distinct covers to begin with. Note that although it may seem that $k = i \mod n$ and $k = n-i \mod n$ will produce equivalent covers, they do not. In the cover with skipping by i, the orientation of the b-cycle will follow that of the a-cycle, and for the other cover, the orientations will be opposing. Further, when k and n are not relatively prime, there will be disjoint b-cycles of order less than n. Regularity requires that each of these b-cycles must agree in orientation with one another. Now, reversing the roles of a and b, we produce another set of n potentially distinct covers. However, by the previous lemma we know for k values such that $gcd(n,k) = 1$, there is only one cover, as interchanging a- and b-edges produces two equivalent covers. Counting the number of relatively prime integers up to n (including 1) with the Euler Φ Function produces the exact number by which we have overcounted. Thus the total number of n-gons with *PS* will be $2n-\Phi(n)$. □

Corollary 4.5. *When n is prime, all regular covers are of the PS class.*

Proof. For a prime n, it is known that there are $n+1$ regular covers. Furthermore, when n is prime, $\Phi(n) = n-1$, so the total number of *PS* covers is $2n-\Phi(n) = 2n-(n-1) = n+1$, by Theorem 4.4. □

4.2. **Double n-gons.**

Definition 4.6. *We will refer to a cover as a* double n-gon *if it is comprised of two nested (n-sided) polygons with sides consisting of, without loss of generality, a-cycles of order n .*

This naturally leads to the question of how the two n-gons (disjoint a-cycles) are connected by b-cycles.

Definition 4.7. *In a double n-gon cover, there is* double n-gon skipping *if* $b^2 = a^k$.

Note that k is allowed to take any value, including zero, and thus every double n-gon can be described with double n-gon skipping.

To allow for a clear representation of double n-gon covers, draw every other edge of a given b-cycle, leaving the other b-edges as implied. This results in a figure consisting of two nested n-gons (where $n = o(a)$), with one directed b-edge between each pair of corresponding vertices. Therefore, given two distinct double n-gon covers with the same b-order, one would not be able to distinguish between them without developing a convention for vertex-labeling.

Lemma 4.8. *Given a double n-gon with skipping by k,* $o(b) = 2r$ *if and only if* $r = \min_{x\in\mathbb{N}}\{xk \equiv 0 \mod n\}$.

Proof. Assume that $r = \min_{x\in\mathbb{N}}\{xk \equiv 0 \mod n\}$. Then r is the minimum number of vertices on the outer n-gon that must be hit before completing a b-cycle. Thus by definition, $o(b) = 2r$. Now let $o(b) = 2r$. This means that we first complete a b-cycle after hitting $2r$ vertices, which by construction means that we hit r vertices of the outer n-gon. Thus we know that $rk \equiv 0 \mod n$. If there existed $m < r$ such that $mk \equiv 0 \mod n$, then moving along m vertices on the outer n-gon, and therefore $2m$ vertices in all, would complete a b-cycle. This would imply $o(b) = 2m < 2r$, which is contradictory to our assumption. Therefore, $r = \min_{x\in\mathbb{N}}\{xk \equiv 0 \mod n\}$. ☐

The following is an immediate corollary to the above.

Corollary 4.9. *In a double n-gon with skipping by k,* $gcd(k,n) = 1$ *if and only if* $o(b)$ *is* $2n$.

It is important to note that in this case, the resulting double n-gon cover is equivalent to the $2n$-gon with *PS* by $2k$, and therefore, when enumerating double n-gon covers, we will only consider those with skipping by values which are not relatively prime to n.

For the sake of computation, it may be convenient to note the following result.

Lemma 4.10. *For any k and n,* $r = \min_{x\in\mathbb{N}}\{xk \equiv 0 \mod n\} = \frac{n}{gcd(n,k)}$.

Proof. Let $gcd(k,n) = y$. We know that $n|rk$ so it must be the case that $\frac{n}{y}|r\frac{k}{y}$. However, $gcd(\frac{n}{y},\frac{k}{y})$=1. Therefore, r must be a multiple of $\frac{n}{y}$. Hence, $\min_{x\in\mathbb{N}}\{xk \equiv 0 \mod n\} = \min_{x\in\mathbb{N}}\{\frac{n}{y}|x\} = \frac{n}{y} = \frac{n}{gcd(n,k)}$. ☐

Lemma 4.11. *In a double n-gon, if* $o(a) = o(b) = n$*, then the two covers with the a- and b-edges interchanged are equivalent.*

To illustrate this statement with an example, consider the case where $k = 2$. Then if n is odd, the smallest r such that $2r \equiv 0 \mod n$ is n, and therefore $o(b) = 2n$. Hence we are only concerned with n being even, in which case the smallest r is $\frac{n}{2}$.

We now label the vertices of the two a-cycles as in Figure 7:

$x_1 - x_2 - \cdots - x_3$

$y_1 - y_2 - \cdots - y_3$

This forces the two b-cycles to be:

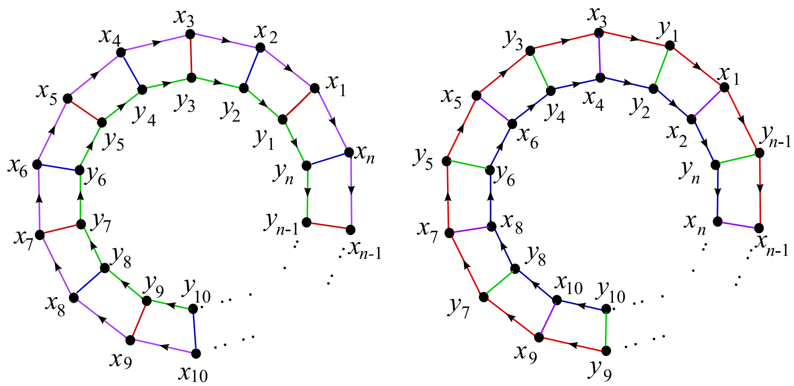

FIGURE 7. Equivalent double n-gon covers with $o(a) = o(b)$

$x_1 - y_1 - x_3 - y_3 - \cdots - x_n$
$x_2 - y_2 - x_4 - \cdots - x_{n-1}$

Redraw the double n-gon so that the two n-gons have sides consisting of b-edges, and align the n-gons such that vertex x_k on the outside corresponds with x_{k+1} on the inside (and therefore y_k and y_{k+1} will maintain the same correspondence). Now adding the appropriate a-edges will yield a double n-gon with skipping by 2, as before and also the same labelings on the a- and b-cycles, by construction.

We can enumerate all possible distinct double n-gon covers.

Proposition 4.12. *The total number of distinct double n-gon covers (excluding those in which $o(b) = 2n$) is $4(n - \Phi(n)) - 2m$ where $m = |\{k < n : \frac{n}{gcd(n,k)} = 2\}|$.*

For each double n-gon with a skipping value k, such that $\frac{n}{gcd(n,k)} = 2$, $o(b) = o(a) = n$, and so by Lemma 4.11 interchanging a- and b-edges yields indistinct covers. Hence for these values, there are only 2 distinct covers: one in which the n-gons (a-cycles) have like orientations, and the other where they have opposing orientations. In all other cases, $o(b) \neq o(a)$, so interchanging a- and b-edges will necessarily yield two distinct covers, as will varying the n-gon orientations, so for these values we have 4 distinct covers.

4.3. **4×2 Covers.** We now consider the infinite family of covers in which $o(a) = 4$ and $o(b) = 2$. Note that by simply interchanging a- and b-edges, we obtain the family of 2×4 covers. As these will have identical properties, we will focus in this section exclusively on the 4×2 case. This family consists of covers with an even number of oriented a-rectangles crossed over each other in sequence as in Figure 8, with b-cycles of length 2 connecting each "neighbouring" pair of vertices not connected with an a-edge. Since the orders of a and b must divide the size of the cover, all covers in this family will be $4m$-fold for some $m \in \mathbb{N}$.

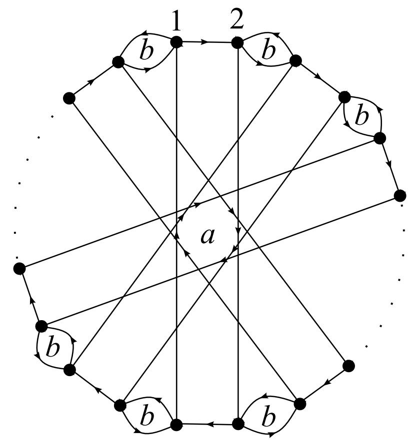

FIGURE 8. A general 4×2 cover with agreeing orientations.

Conjecture 4.13. *In order to maintain regularity, the rectangles in a* 4×2 *cover must be successively crossed over each other as in Figure 8.*

Conjecture 4.14. *There are no regular* $4m$*-fold* 4×2 *covers with opposing orientations where* m *is odd.*

Lemma 4.15. *There are no regular* $4m$*-fold* 4×2 *covers if* m *is odd.*

Proof. Without loss of generality, let a have order 4 and b have order 2. By Conjecture 4.14, we need only concern ourselves with the case in which a-edges are all oriented in the same direction.

We proceed by induction to show that from half of the vertices, $(ab)^m$ is a loop, while from the other half, it is not, thereby proving that the cover is not regular. We will split the vertices into two classes: one, labeled 1, occuring to the right of a b-edge, and another, labeled 2, occuring to the left of a b-edge, as in Figure 8. We will refer to the first class as "1-vertices," and to the second as "2-vertices."

For our base case, we consider the 12-fold 4×2 cover of Figure 9.

This cover is not regular, since $(ab)^3$ is a loop from a 2-vertex, and is not a loop from a 1-vertex.

Now assume that for all odd $k < m$, $(ab)^k$ is a loop from all 2-vertices in the $4k$-fold 4×2 cover, but not from the 1-vertices. We can build the $4m$-fold cover from the $4(m-1)$-fold cover by choosing a b-cycle to "break." Without loss of generality, label the vertices in a clockwise-oriented circle, from v_1 to v_{4m}. Break the b-cycle between vertices v_1 and v_{4m}, which corresponds to breaking the b-cycle between vertices v_2m and v_{2m+1} on the other side. Re-label vertices $v_j \in \{v_{2m+1}, \ldots, v_{4m}\}$ as v_{j+4}, to account for the four new vertices that have been inserted. The four remaining new vertices appear after v_{4m+4} in the sequence, so now there are a total of $4m+8 = 4(m+2)$ vertices. We want to show that $(ab)^{m+2}$ is now a loop from a 2-vertex, and is not from a 1-vertex. Observe that $(ab)^{m+2}$ is not a loop from a 2-vertex, because it traverses only half of the "circle."

To prove that this is a loop from a 1-vertex, we need to show first that once a sequence of ab-edges reaches an edge on one of the two new rectangles, the sequence leaves the new rectangles

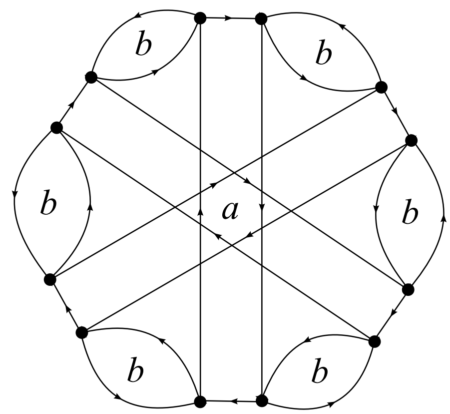

FIGURE 9. A 12-fold 4×2 cover with disagreeing orientations.

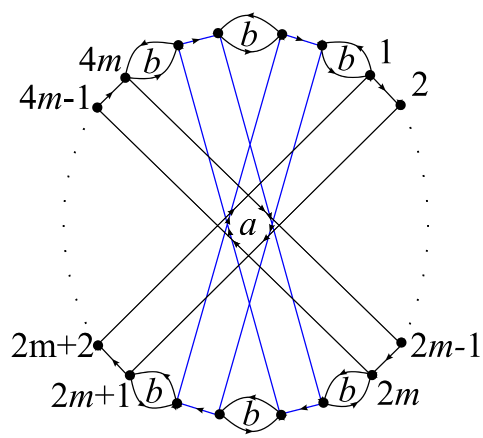

FIGURE 10. Inductive step: inserting two new rectangles into a general 4×2 cover with agreeing orientations.

after going exactly $(ab)^2$. Then we need to show that the vertex at which we entered the new rectangles and the vertex at which we exited correspond to the same vertex if we remove the new rectangles. Supposing that we traveled along $(ab)^i$ before arriving at one of the new edges, we will travel $(ab)^2$ along the new edges, and will exit the new rectangles exactly where we entered. This yields that, since $(ab)^m$ was a loop before, going another $(ab)^{m-i}$ will close this loop. Hence, in the $4(m+2)$ cover, $(ab)^i(ab)^2(ab)^{m-i} = (ab)^{m+2}$ is a loop, as desired.

Note that although there are eight new vertices, there are only four possible vertices at which the sequence can enter the new rectangles, since it is impossible to enter the four vertices which are not adjacent to any pre-existing vertices by a sequence of edges without first touching a new edge. Of these four, two are 1-vertices and two are 2-vertices. Therefore we have only two options for what happens upon arriving to a new rectangle. From both 1-vertices and 2-vertices, $(ab)^2$ consists of all new edges. We can also see that in both of these cases, we enter and exit at corresponding vertices. As described above, $(ab)^{m+2}$ is a loop. Hence, by the principle of mathematical induction, $(ab)^m$ is a loop from a 1-vertex in a $4m$-fold 4×2 cover for all odd $m\in\mathbb{N}$. Therefore, for all odd m, there are no regular $4m$-fold 4×2 covers. □

4.4. **Relating infinite families of covers.** For covers of a given index, it is beneficial to know when the infinite families of section 4 are disjoint. Having such information allows one to certainly avoid double-counting during the enumeration process.

We begin by considering when the aforementioned families may overlap. Given a $4m$-fold cover (where m is some integer), we see that by definition, such covers may be 2×4 covers (having $4m$ vertices), *PS* covers (which may have any number of vertices), or double n-gons (having $2n$ vertices). To begin, the family of 4×2 covers has no cycle (a- or b-) of order greater than 4. However, by definition a $4m$-fold *PS* cover must have a cycle of order $4m$. Thus when $m\neq 1$ we have that these two families must be disjoint. For $m=1$ (a 4-fold cover), 4×2 cover has only one 4-cycle. There is only one 4-cover of this type (a square (without loss of generality) of a-edges whose diagonals are connected by a b-cycle), and it is by definition, not eqivalent to any 4-fold *PS* cover. Thus we have the following lemma:

Lemma 4.16. *A 4×2 cover is not equivalent to a PS cover when both are $4m$-fold covers.*

We now consider the relationship between double n-gons and 4×2 covers. We know by definition that a for a given m value, a double m-gon must have a cycle of order m. However, a 4×2 cover has only cycles of order 4 and 2, and thus for all other integer m values the two families are disjoint. Furthermore, when $m=2$, the corresponding double 2-gon has only cycles of order 2, and therefore cannot be equivalent to a 4×2 cover. Thus the following lemma has been shown:

Lemma 4.17. *A $2m$-fold 4×2 cover is not equivalent to a double m-gon cover when $m\neq 4$.*

Lastly, we must consider when the family of double n-gon covers is disjoint from the family of *PS* covers. For a double n-gon, as stated in section 4.2, when the skipping value k is not coprime to n, there exists no cycle of order $2n$. However, by definition we require a *PS* cover to have at least one cycle of order $2n$. Thus, these two families are disjoint.

Lemma 4.18. *A double n-gon cover with skipping by k, where $gcd(k,n)$, is not equivalent to any $2n$-fold PS cover.*

Thus, in the process enumerating $4m$-fold covers, we have shown that with small number of considerations, the three infinite families are disjoint.

We can now apply this knowledge of infinite families to make an exhaustive list of covers for small index. The following are tables displaying the number of covers of each type for a given fold n. We only give tables for n such that n has more than two numbers in its prime factorization, as the cases when n is prime, or the product of two primes is described extensively in [2].

*Corresponding to the symmetries of the tetrahedron

The totals listed above agree with the number of normal subgroups of given index found by Nieveen and Smith [2, p.115].

a,b order	# Covers	Type of Cover
8×8	4	Planar Skipping
8×4	4	Planar Skipping
8×2	2	Planar Skipping
8×1	2	Planar Skipping
4×4	2	Double Polygon
4×2	4	Disjoint 4-gons
2×2	1	Chain
Total	19	

TABLE 1. 8-Fold Covers

a,b order	# Covers	Type of Cover
12×12	6	Planar Skipping
12×6	4	Planar Skipping
12×4	4	Planar Skipping
12×3	4	Planar Skipping
12×2	2	Planar Skipping
12×1	2	Planar Skipping
6×6	4	Double Polygon
6×4	4	Double Polygon
6×3	-	
6×2	4	Double Polygon
4×4	-	
4×3	4	Stack
4×2	-	
3×3	-	
3×2	2	$A_4{}^*$
2×2	1	Chain
Total	41	

TABLE 2. 12-Fold Covers

5. GENERALIZATIONS TO $F(r)$

Many of our previous results can be generalized to the free group on r generators, $a_1,a_2,\dots,a_r$. In doing so, we address covering spaces of the wedge of r circles. Let the a_i-length of a word be p_i.

Proposition 5.1. *Let $g\in F(r)$ with $p_1,p_2,\dots,p_r$ not all zero, and let n_i be the smallest non-divisor of p_i. There exists $H\triangleleft F$ of index $\min_i\{n_i\}$ such that $g\notin H$.*

Proof. Consider the regular n-fold cover of the wedge of r circles in Figure 11.

Let $n_k=\min_i\{n_i\}$. Consider the simple n-gon corresponding to the group$\langle a_1,a_2,\dots,a_n|\ a_k^n = 1, a_i = 1 \text{ for } i\neq k\rangle$. Fix an arbitrary initial vertex. Consider $g\in F(r)$ and traveling along the corresponding path in the cover, each a_i-edge where $i\neq k$ is a loop and therefore will not affect the

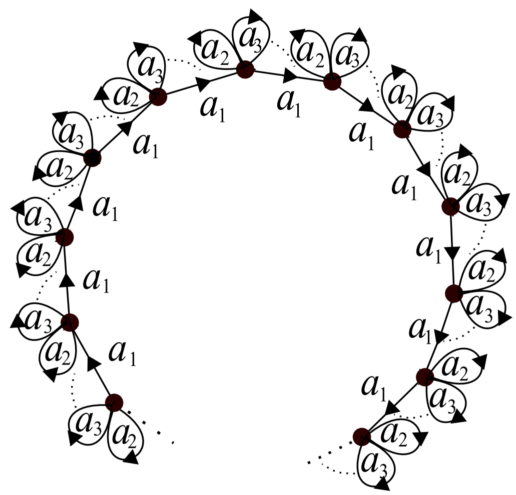

FIGURE 11. Simple n-gon

terminal vertex of the path. Therefore, in considering which words will form a loop in the cover, we need only to consider the a_k edges. We will have an a_k-loop in the simple n-gon if and only if $p_k \equiv 0 \mod n$. Since n_k is the smallest non-divisor of p_k, $p_k \not\equiv 0 \mod n$, and hence the path will not terminate at the initial vertex. Thus g is not contained in the normal subgroup corresponding to the symmetries of the simple n-gon. □

Lemma 5.2. *Given $g \in F(r)$, the symmetry group of the simple n_k-gon is of minimal index among all normal subgroups corresponding to abelian symmetry groups in which g does not lift to a loop.*

Proof. Consider a j-fold regular cover with $j < n_k$ whose symmetry group is abelian, corresponding to a normal subgroup of $F(r)$. Let the j-fold cover have $o(a_i) = m_1$, and let g have a_i-length p_i. We have chosen n_k to be the least non-divisor of all p_i, so we have $j|p_i$ for all $i \leq r$. Furthermore, we know $m_i|j$ [2, pp. 114-115], so we know that there exist $l_i \in \mathbb{Z}$ such that $p_i = l_i m_i$. Note that $\prod_i a_i^{l_i m_i}$ must form a loop, since $\prod_i a_i^{l_i m_i} = \prod_i (a_i^{m_i})^{l_i} = \prod_i 1^{l_i} = 1$ (the trivial word). Because the subgroup corresponding to this regular cover is abelian, any word with a_i-length $l_i m_i$ must form a loop, and thus g lifts to a loop in this cover. Therefore there exists no regular j-fold cover with $j < n_k$ corresponding to an abelian group which does not contain g. □

Naturally, all of the regular covers of the figure-eight space produce regular covers of the wedge of r circles, as any additional generators can be attached to vertices as trivial loops. Given a word $g \in F(2)$ we have been freely choosing a and b to produce a cover in which g is nontrivial. Similarly, given a word $g \in F(r)$ we may choose any two generators a_i, a_j to take the roles of a and b in a cover of the figure-eight space in which g does not lift to a loop. By the simple n-gon construction above, we have produced covers in which g will not lift to a loop for any word $g \notin F(r)'$, with $p_1, p_2, \ldots, p_r$ not all zero. The results enumerating covers of the figure-eight space also apply to covers of the wedge of r circles, but are not exhaustive.

Appendix

Lemma 5.3. *Let* $a,b,c \in \mathbb{N}$ *with* $(a,b) = 1$. *Then* $(a|c$ *and* $b|c) \implies ab|c$.

Proof. $a|c \implies \exists m \in \mathbb{N}$ such that $am = c$, and $b|c \implies \exists n \in \mathbb{N}$ such that $bn = c$. Since $(a,b) = 1$, $\exists$ x,y such that $ax+by = 1$, and so $axc+byc = c$. Therefore $ab(xn+ym) = c$ and $ab|c$, as desired. □

Note 5.4.

It is useful to assert how one may go about showing covering equivalence. Making use of this equivalence, we can describe a process by which one may show the regularity of a given cover.

First, given the wedge of r circles, to show the equivalence of two covers, A with vertices labeled $u_1,\ldots,u_n$ and B with vertices labeled $v_1,\ldots,v_n$, we must first show their equivalence as graphs. Considering only vertices, we must define a homeomorphism $f : A \to B$ (so the covers must have the same number of vertices). Secondly, we must check that if vertex u_i is connected to u_j by a directed q-edge, then $f(u_i)$ is connected to $f(u_j)$ by a directed q-edge (where q refers to a generator in $F(r)$, the free group on r generators). Typically, this can done in the following fashion: first, making use of the labeling of the vertices in A, define a rule which states when two vertices are connected by a given edge corresponding to one of the r generators in $F(r)$. The remaining step is to check if the rules between given vertices in A are preserved by f in their image in B.

It is also important to define a process by which one can show that a given cover of the wedge of r circles is regular. We may do this in terms of the previously defined covering equivalences. For a given cover A of the wedge of r circles, A is regular if there exists a collection of covering equivalences $F = \{f_1,\ldots,f_n\}$ with $f_i : A \to A$, such that for all vertices u_j and u_k in A there exists a composition of elements in F, g, such that $g(u_j) = u_i$.

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