Knight's Tour

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Abstract

Can a knight using legal moves visit every square on a chessboard exactly once? This is a classical problem and the result that there is such a knight's tour for rectangular boards with at least 5 squares on a side seems to be well known. Here, we determine which smaller rectangular boards admit knight's tours. Discovering whether a chessboard has a knight's tour is a more difficult problem if some squares are removed from the chessboard. We show that this is NP-complete by reduction from the Hamiltonian path problem for grid graphs.

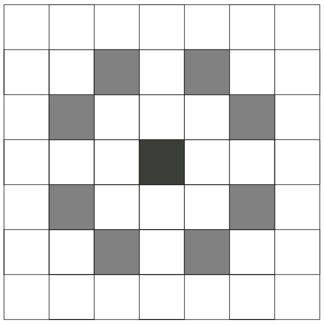
1 Introduction

The knight's tour problem asks whether a knight on a chessboard can visit every square on the board exactly once. The knights circuit requires that the last square on the knight's tour is one legal move from the square that it started on. This problem is a very old chess problem which has been studied by many people. In 1759 Euler demonstrated a knight's tour on an 8×8 chessboard [7]. Results are known for rectangular boards of size greater than 5, but not for smaller boards. It is known how many tours there are on an 8×8 board (there are 33,439,123,484,294) [5], but there is no known general formula for the number of tours on an arbitrary rectangular board.

2 Background

Definition 2.1 *A knight can move to any of 8 different places on a board assuming that it stays within the board. The knight's legal moves are L shaped. She can move from position (i, j) to either $(i \pm 1, j \pm 2)$ or $(i \pm 2, j \pm 1)$.*

Example 2.2 *Here is an example of the knight's possible moves.*



Definition 2.3 *Let a square (i, j) be even if $i + j \equiv 0 \bmod 2$ or odd if $i + j \equiv 1 \bmod 2$, where the upper left corner is square $(1, 1)$. We will call a square's evenness or oddness the color of the square. This corresponds to the red and black coloring of a typical chessboard.*

Remark 2.4 *Knight's moves must alternate between even and odd squares on any chessboard. On a board with an odd number of squares a tour must start on an even square and end on an even square. On a board with an even number of squares a tour must start on one color and end on the other color.*

Theorem 2.5 *There are no circuits on any $n \times m$ board if $n \cdot m$ is odd.*

Proof. This is a well known theorem that was also shown by Cull and DeCurtins [2]. Since the number of squares is odd, by remark 2.4 any path must begin and end on an even square. Thus there can be no circuits. ∎

3 Small Boards

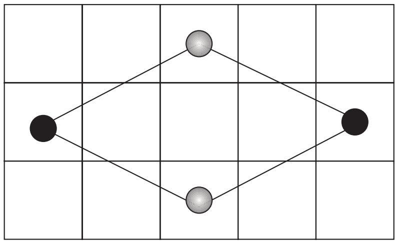
Consider an $n \times m$ rectangular chessboard where $n \leq m$. It has previously been proven by Cull and DeCurtins [2], that if $n \geq 5$ then there is a tour on the board. Additionally, there is a circuit on the board if and only if $n \cdot m$ is even. We completed this result by finding when tours and circuits exist on boards where $n < 5$. Clearly no tours exist on boards with $n \leq 2$. Thus, we consider the cases where $n = 3$ and $n = 4$.

3.1 Boards of Width 3

3.1.1 Tours

Lemma 3.1 *There does not exist a tour of a 3×5 board.*

Proof. Assume that there is a tour. Since the board has an odd number of squares such a tour would need to begin and end on an even square.



3×5 board

The picture above illustrates that the left and right edge squares (the black vertices) have degree 2 and that they share the two middle edge squares (the grey vertices). Therefore you must start or end on a black vertex. Both of these black vertex squares are odd, so neither of them can be an starting or ending square. Now we have a contradiction. Therefore there is no tour on a 3×5 board. ■

Lemma 3.2 *There does not exist a tour of a* 3×6 *board.*

Proof. Exhaustive search. ■

Theorem 3.3 *There exists a tour on a* $3 \times m$ *board unless* $m = 3, 5, 6$.

Proof. The 3×3 case can not have a tour because the middle square is not adjacent to any other squares. In other words, the graph corresponding to the 3×3 case is not connected. The 3×5 case is impossible by lemma 3.1 and so is the 3×6 case by lemma 3.2. We will prove that a tour exists on the remaining $3 \times m$ boards by induction. We will show a tour beginning in the upper left for boards of size $m = 4, 7, 9, 10$. These boards can then be connected together to form all possible $3 \times m$ boards except $m = 3, 5, 6$, all of which were proven to be impossible.

1	4	7	10
8	11	2	5
3	6	9	12

The 3×4 board

1	14	17	20	11	8	5
16	19	12	3	6	21	10
13	2	15	18	9	4	7

The 3×7 board

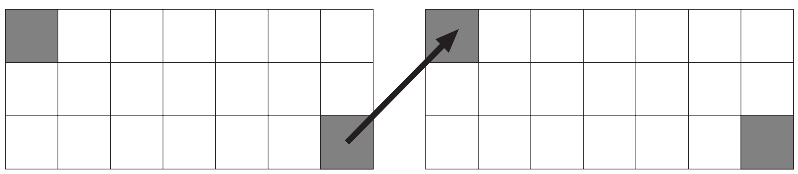
1	14	17	10	7	4	19	22	25
16	9	12	3	18	23	26	5	20
13	2	15	8	11	6	21	24	27

The 3×9 board

1	4	7	22	15	20	13	26	29	18
8	23	2	5	10	25	16	19	12	27
3	6	9	24	21	14	11	28	17	30

The 3×10 board

Given any $3 \times m$ board where $m \neq 3, 5, 6$, the above boards can be strung together where the numbers on the boards are incremented appropriately.



Two 3×7 boards hooked together

■

3.1.2 Circuits

Lemma 3.4 *There are no circuits on the* 3×4*,* 3×6 *or the* 3×8 *board.*

Proof. We will prove in theorem 3.15 that any board with dimension 4 has no circuit, thus the 3×4 board cannot have a circuit. By lemma 3.2, the 3×6 board cannot have tours, thus it cannot have a circuit. The 3×8 board was shown to have no circuits by exhaustive search. ■

Definition 3.5 *An* $n \times m$ *tab board is a board with at least one tour with specific starting and ending squares. The tour begins in the upper left square and ends in a square such that the knight may jump off the board to the position directly to the left of the starting square. These boards are called tab boards because the starting and ending squares can be thought of as "tabs" which will be used in order to connect them to other boards, creating a circuit. The figures below illustrate the tab boards.*

These are the tab boards that we will need for constructing the circuits.

	1	14	17	20	11	8	5
	16	21	12	3	6	19	10
	13	2	15	18	9	4	7

3×7

1	22	3	26	17	20	9	14	11
4	27	24	21	6	15	12	19	8
23	2	5	16	25	18	7	10	13

3×9

1	22	3	6	19	8	25	28	17	14	11
4	33	20	23	26	31	18	9	12	29	16
21	2	5	32	7	24	27	30	15	10	13

3×11

1	30	3	6	33	8	11	36	23	26	15	20	17
4	39	32	29	10	37	34	27	12	21	18	25	14
31	2	5	38	7	28	9	22	35	24	13	16	19

3×13

Definition 3.6 *An extension board is a board which is an overlap of two tours. It is not a tour by itself, but it starts in one corner, exits in the opposite corner, then it comes back in the third corner to exit in the fourth corner having completed all the squares. This is illustrated below in the figure.*

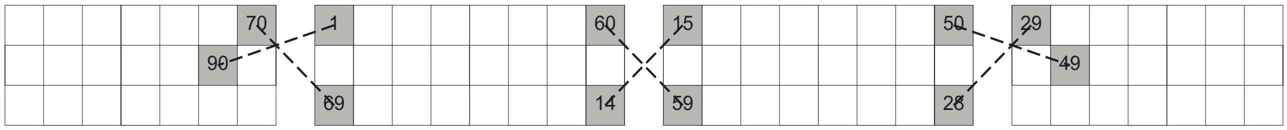
1	6	3	19	22	10	13	15
4	18	23	7	12	16	21	9
24	2	5	17	20	8	11	14

3×8 extension board

The 3×8 extension board below will be used in combination with tab boards to make a circuit board.

Theorem 3.7 *There exists a circuit on a $3 \times m$ board if and only if m is even and $m \geq 10$.*

Proof. As many of these extension boards as necessary may be chained end to end and then terminated at both ends by one of the known $3 \times m$ tab boards. This is illustrated below.



3×30 circuit made with two 3×7 and two 3×8.

With the above construction, the smallest possible board that we can make is a 3×14 by using two 3×7 boards, but any larger board with m even may be constructed. To complete the theorem for $m = 10$ or $m = 12$ we use the circuits below.

1	4	7	16	13	28	11	20	25	22
6	15	2	29	8	17	26	23	10	19
3	30	5	14	27	12	9	18	21	24

3×10 circuit

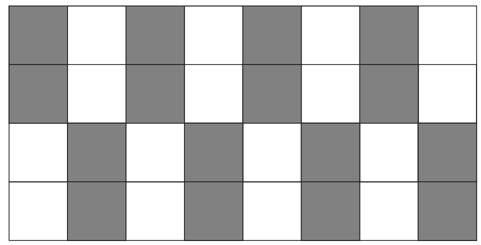
1	34	3	6	9	32	23	26	29	14	19	16
4	7	36	33	22	25	30	11	20	17	28	13
35	2	5	8	31	10	21	24	27	12	15	18

3×12 circuit

By lemma 3.4 there are no circuits for the 3×4, 3×6 or 3×8 boards, and by theorem 2.5 there are no circuits for any $3 \times m$ board where m is odd. This completes the proof. ∎

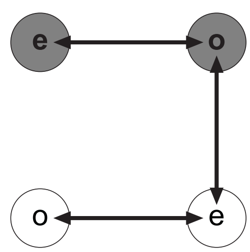
3.2 Boards of Width 4

Notation 3.8 *In order to talk about the $4 \times m$ case, we need to use another parity concept. We will think of the boards as interweaving sets of staggered dominoes. One set of dominoes is grey while the other is white as pictured below. We notice that without loss of generality we can assume that the knight starts on a grey square, because if she started on a white square we could just reflect the board about the horizontal, switching the grey-white coloring.*



4×8 colored board

Remark 3.9 *Notice that you can only jump between a grey square and a white square by jumping between rows 2 and 3. We see that every grey square on rows 2 and 3 is odd and every white square on rows 2 and 3 is even. Therefore, in order to jump from the grey squares to the white squares you must jump from an odd square to an even square. Likewise, to jump from the white squares to the grey squares the knight jumps from an even to an odd square. The allowed moves are illustrated below in a diagram. Each of the four vertices represents a set of squares. They are divided into 4 equal groups: even grey, odd grey, even white, and odd white. On a* $4 \times m$ *board there are* m *elements in each group.*



Theorem 3.10 *Any tour which starts on a grey square must visit all the grey squares before visiting a white square.*

Proof. Consider the diagram in the figure above. Call the even grey vertex v_1, the odd grey vertex v_2, the even white vertex v_3, and the odd white vertex v_4. We will denote the number of times the edge between v_1 and v_2 is used by E_1, and the number of times the v_2v_3 and v_3v_4 edges are used by E_2 and E_3 respectively. We will denote the number of times each v_i is visited by C_i. By remark 3.9 $C_1 = C_2 = C_3 = C_4$. Note that E_1, E_2, $E_3 \in \blacksquare^+$. Now we write down each C_i explicitly by breaking down the diagram into possible cases.

Case 1: Assume that we start on v_2. Then we have $C_1 = \lceil \frac{E_1}{2} \rceil$, $C_2 = 1 + \lfloor \frac{E_1}{2} \rfloor + \lfloor \frac{E_2}{2} \rfloor$, $C_3 = \lceil \frac{E_2}{2} \rceil + \lfloor \frac{E_3}{2} \rfloor$, and $C_4 = \lceil \frac{E_3}{2} \rceil$.

A: Suppose E_1 is even. Then we see that $C_1 = C_2 \Longrightarrow 0 = 1 + \lfloor \frac{E_2}{2} \rfloor$ which simplifies to $E_2 < 0 \rightarrow\leftarrow$ which is a contradiction because all the edges must be taken at least once.

B: Suppose E_1 is odd. Then you must end on v_1, and this forces E_3 to be even. Thus, $C_3 = C_4 \Longrightarrow \lceil \frac{E_2}{2} \rceil + \lfloor \frac{E_3}{2} \rfloor = \lceil \frac{E_3}{2} \rceil$, and because E_3 is even, $\lceil \frac{E_2}{2} \rceil = 0$ which implies that $E_2 = 0$. $\rightarrow\leftarrow$

Since case 1 shows that we cannot start on v_2, by symmetry it also shows that we cannot end on v_2. Also, by symmetry of v_2 and v_3, we cannot start or end on v_3.

Case 2: Assume that we start on v_1. $C_1 = 1 + \lfloor \frac{E_1}{2} \rfloor$, $C_2 = \lceil \frac{E_1}{2} \rceil + \lfloor \frac{E_2}{2} \rfloor$, $C_3 = \lceil \frac{E_2}{2} \rceil + \lfloor \frac{E_3}{2} \rfloor$, and $C_4 = \lceil \frac{E_3}{2} \rceil$.

A: Assume we end on v_1. Therefore E_1,E_2,E_3 are even so the floors and the ceilings can be dropped from the above equations. $C_3 = C_4 \Longrightarrow \frac{E_2}{2} + \frac{E_3}{2} = \frac{E_3}{2} \Longrightarrow E_2 = 0$. $\rightarrow\leftarrow$.

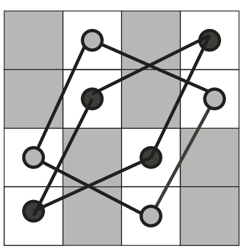
B: Assume we end on v_4. Therefore E_1,E_2,E_3 are odd so $C_1 = 1 + \frac{E_1-1}{2} = \frac{E_1+1}{2}$, $C_2 = \frac{E_1+1}{2} + \frac{E_2-1}{2} = \frac{E_1+E_2}{2}$, $C_3 = \frac{E_2+1}{2} + \frac{E_3-1}{2} = \frac{E_2+E_3}{2}$, and $C_4 = \frac{E_3+1}{2}$. $C_1 = C_2 \Longrightarrow E_1 = E_3$ and $C_3 = C_4 \Longrightarrow E_2 = 1$.

This is what we wanted to prove. You must start on v_1 and the edge between v_2 and v_3 must be used exactly once. It is clear that this one use of the edge between v_2 and v_3 must occur after the counts of v_1 and v_2 are completed. In other words, you must tour all the grey squares and then tour all the white squares. ■

3.2.1 Tours

Lemma 3.11 *There is no tour on a 4×4 board.*

Proof. Because the 4×4 board is symmetric, we can assume that we start on a grey square. Using theorem 3.10 we need to complete all the grey squares before jumping to a white square. Imagine that a tour had finished the grey squares. We will show that from any position on the white squares there cannot be a tour. This is because there are two disjoint cycles. Since there is no possible way to jump from one cycle to the next, there are always four white squares left over.

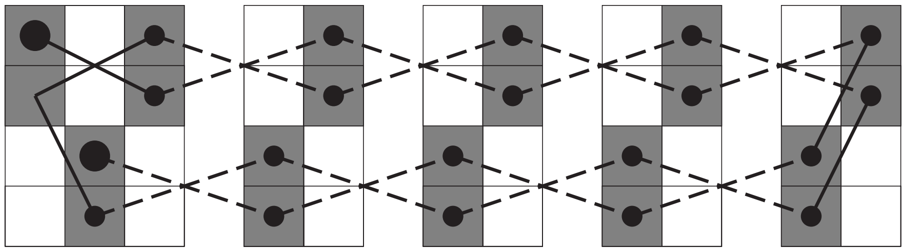


4×4 board

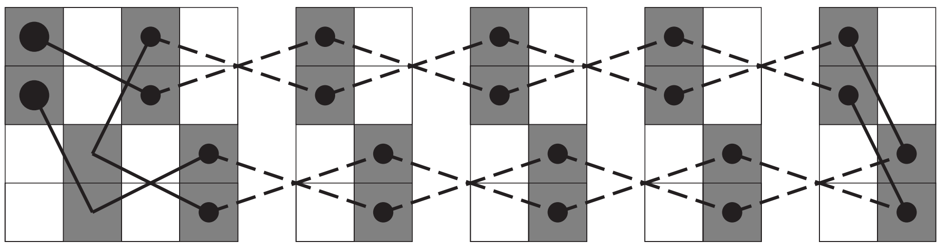
■

Lemma 3.12 *The grey squares can be toured on a $4 \times m$ board for $m \geq 5$.*

Proof. To construct a tour of the grey squares, we use 1 beginning board, followed by 0 or more middle boards, and finally 1 ending board. Then all the sub-boards are connected as shown in the figure. The tour begins in the upper-left and the square that the tour finishes on is determined by whether m is even or odd.



Odd Case

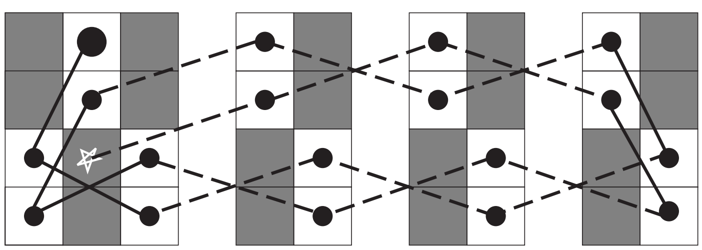


Even Case

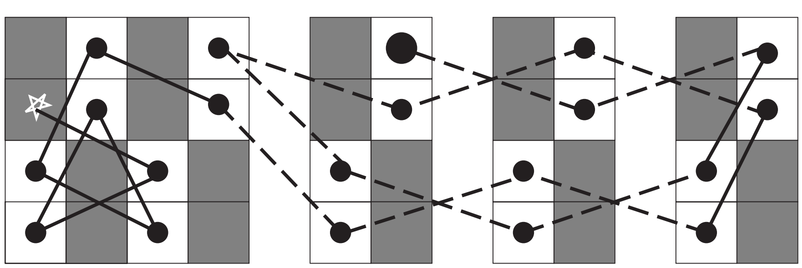
■

Lemma 3.13 *A tour of the white squares can be completed on a $4 \times m$ board for $m \geq 5$, beginning in one of two squares.*

Proof. As above, there is a beginning board, followed by 0 or more middle boards, and finally 1 ending board.



Odd case



Even case

■

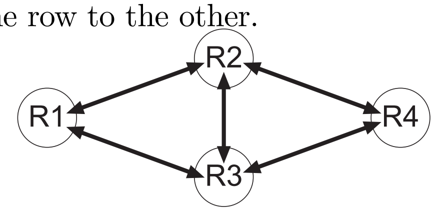
Theorem 3.14 *There is a knight's tour on $4 \times m$ boards for all $m > 4$.*

Proof. A tour on any $4 \times m$ board can be completed by the previous lemmas. First do all the grey squares. In either case, the knight can jump from the grey finishing square to the beginning of the tour of the white squares. This covers every square exactly once, so it is a tour. This process will work for $m \geq 5$. This covers all cases of boards of size $4 \times m$ except $m = 4$ on which there is no tour by lemma 3.11. ■

3.2.2 Circuits

Theorem 3.15 *For boards of size $4 \times m$ there are no circuits.*

Proof. Consider the following diagram. It illustrates which rows are accessible from which other rows. There is an edge between two vertices if a legal move will take you from one row to the other.



Since a circuit can start from anywhere, assume the knight starts on row 1 (move 1 is on row 1). We note that the number of squares (and thus the number of moves) is even. If we visit an even number of vertices, but never take the edge between R2 and R3, then we will end on R2 or R3 (by inspection). But by theorem 3.10 and remark 3.9 we must take the edge between R2 and R3 exactly once, so we must end on R1 or R4. But we started on R1, so there is no way we can make one more move and get back there! Thus there are no circuits. ■

4 NP-Completeness

Definition 4.1 *A decision problem is a problem that has a yes or no answer.*

Definition 4.2 *A decision problem is in NP (non-deterministic polynomial time) if given a particular problem and a potential solution, a polynomial time algorithm can verify that the solution is correct.*

Example 4.3 *The Hamiltonian Path Problem is in NP because given a graph and a path of vertices, it is easy to verify that the path visits every vertex exactly once and that between any two consecutive vertices there is an edge.*

Definition 4.4 *A problem in NP is NP-complete if every problem in NP is polynomial time reducible to that problem, meaning no problem in NP requires a longer asymptotic running time. Two things are required in order to prove that a problem is NP-complete. The problem must be proven to be in NP and it must be shown that another problem which is already known to be NP-complete reduces to that problem. For a further description of the theory of NP-completeness, see Cormen et al [1]. For an extensive list of known NP-complete problems see Garey and Johnson [3].*

Definition 4.5 *A board with holes is a chessboard where any number of individual squares have been removed.*

Definition 4.6 *A grid graph is graph on the plane where all vertices have integer coordinates. An edge can only connect* 2 *vertices if the Euclidean distance between them equals* 1*.*

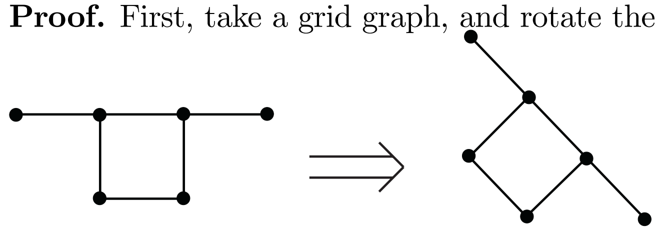
4.1 Knight's Tour on Boards With Holes

Lemma 4.7 *The knight's tour problem with holes is in NP.*

Proof. Given a chessboard with holes and a list of nodes it can be determined in polynomial time if all nodes are listed exactly once, and that there is a legal knight's move from one node to the next. ■

Lemma 4.8 *We can construct in polynomial time a chessboard with holes from a grid graph. (We will use this to reduce Hamiltonian paths on grid graphs to the knight's tour on boards with holes.)*

Proof. First, take a grid graph, and rotate the graph 45° clockwise.



Next, overlay the grid graph with a chessboard such that each node is covered by a 5×5 board with one row between all the boards horizontally, and no rows between the boards vertically. This way the knight can jump from the corner of one 5×5 to the corner of the next, but there are no other paths between the boards. This corresponds to the grid graph containing every possible edge. See figure A. To represent the case where two nodes should not be connected by an edge, we remove two squares from the appropriate corner of each of the two nodes. If there is only one edge emanating from a node, then the node is represented as just the corner square of the 5×5 board, as seen below in figure B. If there are zero edges emanating from a node, then the grid graph is not connected and the isolated node can be represented by one square in the center of the 5×5 board. (This would make the knight's tour impossible unless it was the grid graph with one vertex.)

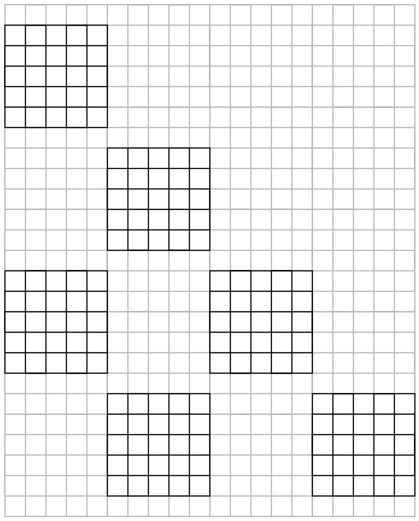


Figure A

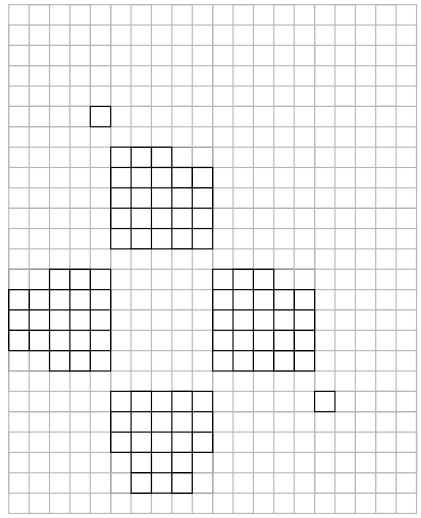


Figure B

This construction can be done in polynomial time. ■

Theorem 4.9 *If there is a Hamiltonian path on the grid graph, then there is a knight's tour on the constructed chessboard.*

Proof. Given a grid graph and a Hamiltonian path, construct the chessboard. The knight's tour follows the Hamiltonian path on the grid graph. Using the boards below, tour each node starting in one corner and leaving through another. In this way the whole board can be toured in the order given by the Hamiltonian path on the grid graph.

1	18	7	12	25
8	13	24	17	22
19	2	21	6	11
14	9	4	23	16
3	20	15	10	5

1	14	21	8	3
20	9	2	13	18
15	22	19	4	7
10	5	24	17	12
23	16	11	6	25

1	6	11		
12	19	14	5	10
7	2	17	20	15
18	13	22	9	4
23	8	3	16	21

1	6	11		
12	21	14	5	10
7	2	17	22	15
20	13	4	9	18
3	8	19	16	23

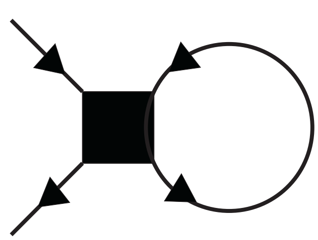
1	10	15		
16	5	18	9	14
11	2	7	4	19
6	17	20	13	8
21	12	3		

1	10	15		
16	5	12	9	14
11	2	7	4	19
6	17	20	13	8
		3	18	21

Touring individual nodes

Every possibility is some reflection or rotation of the above tours. ■

Lemma 4.10 *It is impossible to pass through any node twice as shown below.*



impossible path

Proof. Obviously the 5×5 node is only one we need to consider, since no other node has degree 4. Whenever the knight enters or leaves the 5×5 node, she must use an even square. Therefore every time the knight passes through the node, the number of even squares she uses must be exactly one more than the number of odd squares she uses. With two passes through the node, the number of even squares she uses is two squares greater than the number of odd squares. This makes two passes through the node impossible. ■

Lemma 4.11 *The only nodes that the knight's tour can visit twice are the starting and ending nodes. Ignoring the case where the starting and ending nodes are the same we also have the following. The starting node is used twice if and only if the knight starts on an odd square. Similarly, the ending node is used twice if and only if she ends on an odd square.*

Proof. By lemma 4.10 the knight cannot pass through a node twice. Thus if she visits any node twice she must start or end there. ($\Longrightarrow$) We will provide the argument for the starting node and the corresponding argument for the ending node. (1) Suppose the starting node was used twice in a successful tour. Since we are assuming that the starting node is not the ending node, the knight would have to pass completely through the node on her second visit. This forces the knight to start on an odd square. Otherwise passing through it would be impossible by the same reasoning in lemma 4.10. (2) Suppose the ending node was used twice in a successful tour. Since the ending node is not the starting node, the knight must have entered the node on an even square during the first visit. This forces her to end on an odd square on her second visit to the node. ($\Longleftarrow$) Now we show the converse of both cases. (1) Suppose the knight starts on an odd square. Since she must leave through an even square, the number of even squares remaining on the node must be precisely one more than the number of remaining odd squares. Thus the knight is forced to return to the node in order to tour every square. (2) Since the starting and ending nodes are different by assumption, the knight must enter the ending node through an even square. Now suppose the knight finishes on an odd square. This forces the number of even squares and odd squares toured in her last visit to be equal. But we know the number of even squares is greater by 1, so in order to have completed the node, she must have been there before. This completes the proof. ■

Theorem 4.12 *If there is a knight's tour on the constructed chessboard, then there is a Hamiltonian path on the grid graph.*

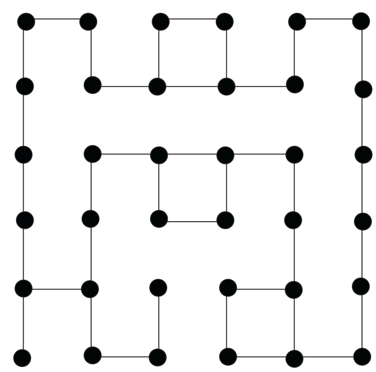
Proof. Having a knight's tour of the constructed chessboard certainly implies that a path on the grid graph exists where every vertex is visited at least once. However, we need a path where every vertex is visited exactly once. By lemma 4.11 the only nodes we could have visited twice are the starting and ending ones, so those are the only ones we need to consider. If the starting and ending nodes are the same we have a hamiltonian cycle on the grid graph, so we certainly have a hamiltonian path. Now we just need to look at the case were the starting node is visited twice or the ending node is visited twice. We claim that there is a simple method of translating such a knight's tour into a Hamiltonian path on the grid graph. To translate a given knight's tour into a hamiltonian path on the grid graph, first look at the sequence of nodes given by the tour, for example $v_1, v_2, ..., v_n$, if the tour of node v_1 starts on an odd square, then it must be repeated later on in the sequence by lemma 4.11 so remove it from the sequence leaving, $v_2, ..., v_n$ as the new sequence. Similarly, if the tour of v_n ends on an odd square, it must have been visited earlier in the tour by lemma

4.11 so remove it. The remaining list of nodes will create a hamiltonian path on the grid graph. ■

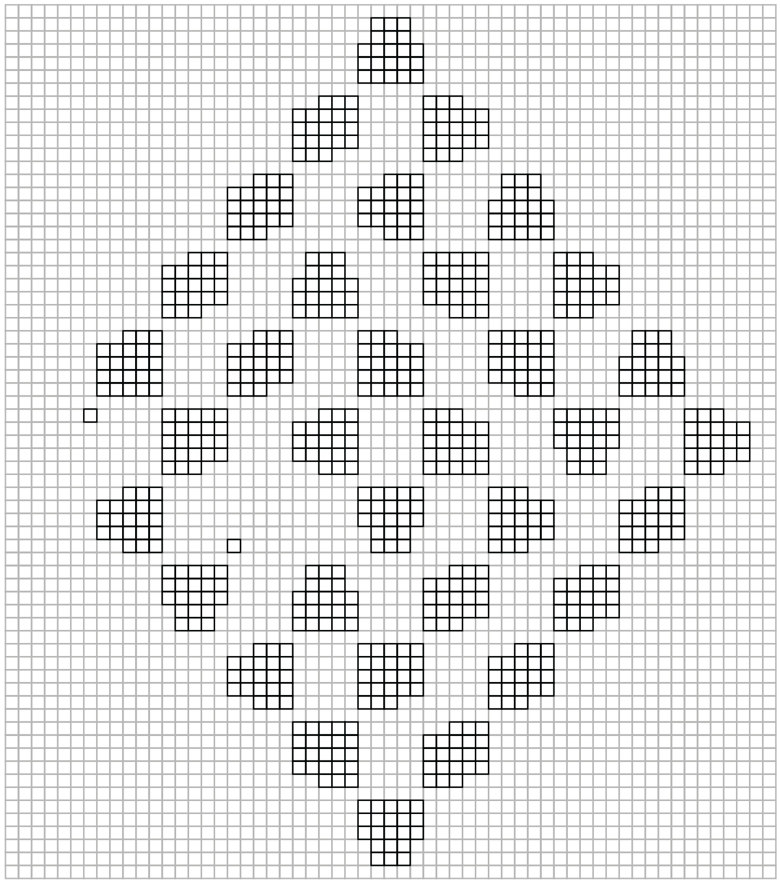
Theorem 4.13 *The knight's tour problem with holes is NP-complete.*

Proof. By the lemma 4.7, the knight's tour problem with holes is in NP. Also, by lemma 4.9 and lemma 4.12, the construction given in lemma 4.8 reduces Hamiltonian paths on grid graphs to the knight's tour problem with holes. It has been shown by Itai et al [4] that Hamiltonian paths on grid graphs are NP-complete. Therefore, the knight's tour problem on a chessboard with holes is also NP-complete. ■

Example 4.14 *Here's an example of a potential grid graph. There happens to be exactly one hamiltonian path on this grid graph. Finding this path is left as an exercise to the reader.*



Below is the chessboard constructed based on the above grid graph. Notice that the hamiltonian path corresponds to a knight's tour on the chessboard. Also notice that any knight's tour one can find on this chessboard corresponds to the path on the grid graph.



4.2 Knight's Tour on Connected Boards With Holes

Definition 4.15 *We call a chessboard with holes connected if the rook could move from any square to any other in as many turns as necessary. More technically, consider the board as a graph where every square is a vertex and two squares are connected by an edge if and only if they share a side. Then a connected chessboard would just be a chessboard where this graph is connected.*

Problem 4.16 *After seeing the construction, the natural question seems to be whether the knight's tour problem on connected boards with holes is still NP-complete. We originally conjectured that this was the case. However, when we tried connecting the nodes in our construction with various strange shaped boards representing edges, we could not find any "edge boards" for which theorem 4.12 was true. After awhile it seemed to appear that this same approach would not work for connected boards. Although it seems most likely that knight's tour on a connected board with holes is still NP-complete, it is possible that some algorithm could exist which would tell us whether a connected board with holes had a tour. It would be nice to settle this problem one way or another.*

5 Conclusion

We successfully showed which rectangular chessboards have knight's tours and which have circuits. After doing this we found a previous paper by Schwenk [6] that did the same thing just for circuits. We left our approach in the paper because it was different from Schwenk's. Our results for the smaller boards combined with Cull's previous result [2] show that the knight's tour problem on rectangular boards is in P. In contrast, we have shown that the knight's tour with holes is NP-complete. Although we suspect that the problem would remain NP-complete if the boards remain connected, we were not able to prove it. This could be an interesting topic to continue work on. Also it would be interesting to look into #P-completeness of rectangular boards. We suspect that there is not an easy algorithm for computing the number of tours on a rectangular board, and that the problem is in fact counting hard. It would be interesting to prove or disprove this conjecture.

References

[1] T. Cormen, C. Leiserson, and R. Rivest. *Introduction to Algorithms*. MIT Press. Cambridge, Massachusetts, 2000.

[2] P. Cull and J. De Curtins. "Knight's Tour Revisited." *Fibonacci Quarterly*. vol. 16, pp. 276-285. June, 1978.

[3] M. Garey and D. Johnson. *Computers and Intractability, A Guide to the Theory of NP-Completeness*. W.H. Freeman and Company. San Francisco, 1979.

[4] A. Itai, C. Papadimitriou, and J. Szwarcfiter. "Hamiltonian Paths in Grid Graphs." *SIAM Journal Computation*. vol. 11, pp. 676- 686. November, 1982.

[5] M. Lobbing and I Wegener. "The Number of Knight's Tours Equals 33,439,123,484,294 - Counting with Binary Decision Diagrams." ftp://ftp.eccc.uni-trier.de/pub/eccc/reports/1995/TR95-047/index.html

[6] Schwenk, Allen. "Which Rectangular Chessboards Have a Knight's Tour?." *Mathematics Magazine*. vol. 64, pp. 325-332. December, 1991.

[7] K. Seibel. "The Knight's Tour on the Cylinder and Torus." REU at Oregon State University. August, 1994.