THE KNIGHT'S TOUR ON THE CYLINDER AND TORUS

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ABSTRACT

By mapping the chessboard onto the cylinder and the torus, a much greater range of motion is allowed on that board. Using the same legal knight's move as on the chessboard, I have shown that on the torus, a knight's circuit is possible for any size board. I demonstrate these constructively and hence give a fast algorithm to find these circuits. The situation is slightly more complicates on the cylinder. I give a complete description of those cylinders which have knight's circuit and those which have knight's tours, but no circuits. Again, the methods I used are constructive and lend to fast algorithms. I have also summarized known results about knight's circuits and tour on the rectangular chessboard.

INTRODUCTION

The knight's tour problem has been around for a long time. Back in 1759, Euler tried the first mathematical analysis of the problem [Eu]. The problem started out as a puzzle on the regular 8*x*8 chessboard. Using only the legal knight's move, could one move the knight around the board so that it visits each square exactly once and only once and returns to the starting square? A legal move consists of moving two squares horizontally and one square vertically, or two squares vertically and one square horizontally. If the final square is a legal move from the starting square, then the tour is called a closed tour or a knight's circuit. When the starting square cannot be reached from the final numbered square, the resulting tour is called an open tour of the chessboard, or a knight's tour. As time progressed, the puzzle was extended to chessboards of other sizes and the attempt has been to determine which boards contain a circuit or tour. In more recent years, with the growth of the computer revolution, the emphasis has been on finding computer algorithms which are efficient at finding a knight's tour on any board.

There has also been some work, lately, with regard to finding knight's tours on boards with a square missing. After reading Hurd and Trautman's paper [Hu] on the knight's tour of the 15-puzzle, I began to wonder which deformations of the chessboard would yield a knight's tour or circuit. After playing various deformations, I decided to try mapping the chessboard onto the cylinder and torus. Surprisingly, I could easily find, not only a tour, but a circuit on most cylinders and tori.

To avoid confusion, I created a system, in this paper, for noting the cases of boards, cylinders, and tori. To begin with, I have used rectangular representation for all the cylinders and tori. In this paper, an m*x*n board is m units tall and n units wide. The m*x*n cylinder, in rectangular representation, follows the same format as the board, with the m-unit sides identified. To create the m*x*n torus, use the same format as the cylinder and identify the n-unit sides. With this system, note that, for the board and the torus, m*x*n is equivalent to n*x*m. However, this is not the case for the cylinder. Hence, an m*x*n cylinder is distinct from an n*x*m cylinder.

On the chessboard, the traditional bicoloring has been used for many proofs of tours and circuits on the board. It also helped to define a legal knight's move. With the identification of the edges in the cylinder and the torus, the bicoloring of the boards does not always match. So, except for cases where the bicoloring of the squares is useful for a proof or for illustrating an algorithm, it has been disregarded.

Because of the symmetry of the cylinder and torus, many of the algorithms discussed here can be flipped top to bottom or left to right to cover other cases than those used. Since the main idea behind the paper was to show that a circuit or tour was possible, I did not go into all the tours and circuits possible for each case.

THE 1 *X* N CASE

For the 1*x*n case, the board and the cylinder make the job easy. Because of extremely limited mobility, it is easily seen that only the trivial case of 1*x*1 has a solution. On the 1*x*n board (see fig. 1) and the 1*x*n cylinder, there is only movement allowed in one direction, horizontally. Since a legal knight's move starting horizontally requires a vertical move to end with, there are no possible moves from the starting square.

Figure 1. The 1*x*N Board

In contrast, mapping the board onto the torus allows for much greater mobility. There are now two directions open for movement. The following algorithm not only gives a knight's tour on the torus, but finds a knight's circuit.

Start at any square. By moving one square horizontally and two vertically, the next number will be placed on the square directly next to the starting square. Repeat this step, moving in the same horizontal direction each time, until the squares are numbered around the torus. This procedure takes the numbering to the square before the starting one. The next step will then move onto square one, i.e., a knight's circuit is complete. See the figure below for the sequence of move. Note that in this figure, and in all following figures, double lines indicate sides which are identified.

This algorithm works for any 1*x*n torus with n = 1 ,2,...(see fig. 2).

Figure 2. The 1*x*N Torus with Knight's Moves

THE 2 *X* N CASE

The 2*x*n case begins to differentiate between the board and the cylinder. Because of the second row that has been added, it is now possible to make a legal knight's move on the board if there are at least three columns. Thus, it is necessary to look at all three shapes separately.

First, I have the following proof that there cannot exist a knight's tour on any 2*x*n board. Number the columns from left to right starting at the left-hand side with zero. Now let the first square be one of the two squares in column zero. Because of the limited mobility, the only move open is to move two columns to the right, ending up in the opposite row in which the numbering started. Continuing in this manner, one square in each even numbered column will be filled.

Figure 3. The 2*x*N Board with Forced Moves

There is no move to reach the odd numbered columns, nor can the other square in each column be numbered (see fig. 3). Hence, a tour cannot be completed.

A similar argument can be used for the 2*x*n cylinder, with n even. Because there is an even number of columns, the alternate numbering of odd and even columns is maintained when the 2-unit sides are identified. A tour still cannot be completed as there is no legal knight's move to reach the odd numbered columns.

When n is odd however, it becomes possible to reach all squares since the moves are not limited to only the even numbered columns. It can be proved there exists a circuit by construction.

The following is an algorithm for finding a circuit on the 2*x*odd cylinder. To find the circuit, start numbering the squares by picking a square in the bottom row as square one and noting that column as column one. Now begin to move around the cylinder in the right hand direction. (It is possible to reverse the cylinder to take care of the case of left hand movement.) When the numbering comes back around the cylinder, there are two cases of where the next move will land before going past the starting square. Because of odd number of columns, the first round of moves ends in column n, the one directly before the first column. The first case of where the move will land is in the same row in which the tour began. These are the cylinders of the form 2*x*(4n+1), n = 0 ,1,2, ... The second case is that the move will end up in the opposite row from the starting square. Here the cylinders are of the form 2*x*(4n+3), n = 0 ,1,2,...

Now the algorithm should be repeated until all squares are filled. In the first case, each repeat of the algorithm starts with a numbered square in the opposite row as the previous iteration and ends in the column before the first square of the repetition and in the same row (see fig. 4 and 5).

		k-1	j		m	
	j-1	1	m-1	k		

Figure 4. Numbering Algorithm for $2x(4n+1)$ Cylinder

	j-1	k-1	m-1			
		1	j	k	m	

Figure 5. Numbering Algorithm for $2x(4n+3)$ Cylinder

In the second case, each repetition ends in the column before the first square of the iteration and in the opposite row. In the above figures, the cycles begin with the squares numbered 1, j, k, and m, in that order.

This algorithm splits the cylinder into four groups of squares, one for each repetition of the algorithm. Notice that filling in the final square of the last iteration allows the tour to close. Hence, by construction, a circuit on the $2x$odd cylinder is complete. Figure 6, below, gives an example of these types of cylinders. The cylinder on the left is the $2x5$ cylinder, while the one on the right is the $2x7$ cylinder.

10	8	6	4	2
5	3	1	9	7

14	4	8	12	2	6	10
7	11	1	5	9	13	3

Figure 6. Examples of $2x$Odd Cylinders.

Now, it is necessary to look at the $2xn$ torus. Since there is already an algorithm for a circuit on the $2x$odd cylinder, half of the job is done. The $2x$odd torus is the $2x$odd cylinder with the n-unit sides identified. Hence, an algorithm for a circuit on the $2x$odd torus exists. Therefore, only the $2x$even torus needs to be considered. Moreover, the following algorithm produces a circuit on not only the $2x$even torus, but also the $2x$odd torus.

The first step is to pick a starting square on the bottom row. Of course, the torus can be renumbered to begin on any square. Begin to number along the bottom row to the right. Each of these moves is a knight's move of one square to the right and two squares upward (see fig. 7). Because of the small diameter of the torus, each successive number is wrapped around to the square next to the one previous.

		n+2	n+1	2n	
	n	1	2		

Figure 7. The Numbering Algorithm for the $2xN$ Torus

The last empty square on the bottom will be numbered n. The next move is to go one square up and two to the right. Now repeat the same type of move as on the bottom row, except move to the left. The last empty square in that row will be 2n, a legal knight's move from square one.

THE 3 *x* N CASE

This is the first case of the board where tours and circuits are possible. It is obvious that not every board will have a circuit or even a tour. Yet, it is within reach to begin to eliminate those cases. By previous arguments, the 3*x*1 and 3*x*2 boards cannot contain a tour. It can easily be seen that 3*x*3 can also be eliminated.

3	8	5
6		2
1	4	7

Figure 8. The Incomplete 3*x*3 Tour

Although all outside squares can be reached, the inner square has no legal knight's move leading to it. This leaves the cases of 3*x*4 and greater.

The first observation is that no 3*x*n chessboard, with n odd, has a knight's circuit. This can be seen by looking at the normal coloring of a chessboard. Because both 3 and n are odd, there will be an odd number of total squares. This means that one color is more numerous than the other. If the tour starts on a least numerous color, those squares will be used up before the others, as a legal knight's move requires alternating between colors. Thus, a square of the most numerous color will be left unnumbered, and hence no tour. The other case is to start on a square of the most numerous color. In this method, the tour will end on a square of the same color it began on, and there is no move to return to the starting square, i.e., no circuit.

John Van Rees, of the University of Manitoba, has done some work with 3*x*n chessboards [Va]. His results are that a 3*x*n chessboard has a knight's circuit precisely when $n=0,2 \bmod 4$ and $n>8$. Van Rees uses that fact that each circuit must use the below partial circuits on the ends of the board.

x	x	x		
x		x	e	
x	x	e		

		x	x	x
	e	x		x
		e	x	x

Figure 9. Van Rees' Partial Circuits

In figure 9, e represents an end of the partial circuit, and x represents squares that must be used in the partial circuit. It is easily shown that for $n<8$, the two sides overlap and these cases are then ruled out. For $n=8$, placing these ends into the 3*x*8 board shows that the partial circuits cannot be linked and the blank squares cannot be filled in a complete circuit of the board. Van Rees claims that a circuit on the 3*x*10 and 3*x*12 boards are easy to find. By rearranging the numbers of some such circuit, it is possible to arrive at a circuit with the right-hand side looking like the below figure.

-----	x	x	2
-----	1	x	x
-----	x	3	x

Figure 10. Right-Hand edge of 3*x*10 or 3*x*12 Circuit

Since a circuit on the 3*x*4 board can be found, it is only needed to find one with the ends in the proper places; see figure 11 below.

-----	x	x	2	-----	3'	6'	11'	8'
-----	1	x	x	-----	12'	9'	2'	5'
-----	x	3	x	-----	1'	4'	7'	10'

Figure 11. The Attachment of a 3*x*4 Tour to the End of Circuit

Now, instead of going from 1 to 2 to 3, go from 1 to 2 to the lower left square of the 3*x*4 tour (1'). The end of the 3*x*4 tour, (12'), then reaches 3 in the base circuit. Since, by the example shown, a 3*x*4 board with the required order can be found, any number of 3*x*4 tours can be added to the base circuit. Only a reordering of the numbers is required to make the new circuit look like the base circuit above. Now a proof of the existence of a circuit for the above cases is complete.

Van Rees also reported that knight's tours occur in the remaining cases when $n=1,3$ mod 4 and $n \geq 7$ and when $n=4$. Using a similar 3*x*4 tour, apply the same type of proof as above to this problem. The below is an example of a 3*x*4 tour which satisfies Van Rees' requirements that the tour begin in one corner and end in the diagonally opposite corner.

3	6	9	12
8	11	2	5
1	4	7	10

Figure 12. 3*x*4 Tour with Diagonally Opposite Ends

He now claims it is easy to find tours on the 3*x*7 and 3*x*9 boards which end as shown below in figure 13.

-----	x	x	x	x
-----	x	x	e	x
-----	x	x	x	x

Figure 13. Right-Hand Edge of 3*x*7 or 3*x*9 Tour

To extend the 3*x*7 tour, in figure 14, to the 3*x*4 tour, make the move from 21 to 1' and complete the smaller tour. It is possible to add the 3*x*4 tour to the end of the base tour and to itself. Hence, the case stated above are proved to have tours.

7	10	5	2	19	16	13	3'	6'	9'	12'
4	1	8	11	14	21	18	8'	11'	2'	5'
9	6	3	20	17	12	15	1'	4'	7'	10'

Figure 14. Example of Attachment of 3*x*7 to 3*x*4

In his paper, Schwenk also gives the same case of exemption for circuits on the 3*x*n board [Sc]. He states the following cases as not having a circuit; the case of 3*x*odd and the cases 3*x*(4,6,8). This agrees with Van Rees' findings.

The 3*x*n cylinder is the next case to consider. There is an easy algorithm to find a circuit on the 3*x*n cylinder. Like the above circuit algorithm for the 2*x*n torus, this algorithm does not differentiate between even and odd n's, and works easily for either one. Also, this algorithm uses one square in each row for each iteration. This allows the addition of a square in each row, in the appropriate place, to move up to the 3*x*(n+1) cylinder.

The algorithm begins by choosing a square in the bottom row as a starting square. From square one, move one square right and two squares upward. Then move one square down and two squares to the left. This is the first iteration. The next step goes to the starting square of iteration two. Move one square down and two squares to the right. This move advances to the first square of the next repetition. Notice that this is one square to the right of square one. Figure 15 shows one repetition of the algorithm with the moves a, b, and then c.

		b
c		
	a	

Figure 15. Repetition in the 3*x*N Algorithm

This algorithm allows movement one square at a time down each row. This means the circuit simply wraps around the cylinder until it matches up to the starting point. After reaching the last repeat of the algorithm, the final square will be a legal knight's move away from square one (see fig. 16). Thus the circuit is complete.

		3n-1	2	5
3n	3	6		
	3n-2	1	4	7

Figure 16. The 3*x*N Circuit Wrapping Around the Cylinder

The final case to consider is the torus. Because there exists an algorithm for the 3*x*n cylinder, the next step is to identify the n-unit sides of the cylinder to create the torus. The previous algorithm works just as nicely on the torus.

This algorithm has some nice qualities that can be extended to other sizes of cylinders and tori. The first, and most helpful of these qualities, is that the algorithm uses one square in each row for each repetition. This allows for expansion of the cylinder or torus by one unit or by several. The new squares only need to be added to an already existing cylinder in the below order. Thus no renumbering needs to be done. Figure 17 shows an example of blocks being added to a 3*x*4 circuit to extend it to a 3*x*5 circuit.

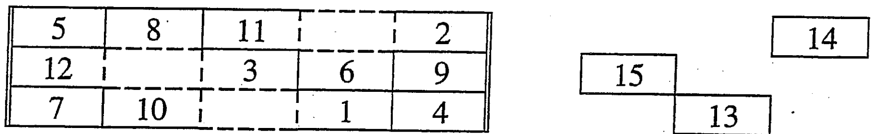

Figure 17. Blocks to be Added to 3*x*4 Circuit to Make 3*x*5 Circuit

The second nice quality of this method is that the starting square of each repetition is a legal knight's move from the last square of the previous repetition. In this manner, the numbering wraps around the cylinder or torus, filling in each square as it goes and does not rely on repeated cycles around the piece, as does the algorithm for the 2*x*odd cylinder. There are no empty squares left behind. These two qualities helped to construct later algorithms for larger cylinders and tori.

THE 4 *X* N CASE

The 4*x*n board seems to have more restrictions than the 3*x*n board. Although the board was not the main object of my research, I did find a few conditions for this case reported by other researchers. Both Schwenk [Sc] and Eggleton and Eid [Eg] give proofs for the statements that no 4*x*n board contains a knight's circuit. Obviously, this is much more restrictive than the 3*x*n board which did have circuits on some boards.

As for tours on the 4*x*n board, I did not go into that very much. From previous arguments, it is known that 4*x*(1,2) do not have tours, while 4*x*3 does have a tour. It is known that the 4*x*4 board does not have a tour. It is conceivable that a method similar to that used be Van Rees could be used to create larger tours. If a 4*x*n board has a tour, it may be possible to create a 4*x*(n+3) board with a tour.

For the 4*x*odd cylinder, I found an algorithm that finds a circuit using a similar method to that on the 3*x*n cylinder. Unfortunately, this method does not construct the algorithm as cleanly. Here, one square in each row is used for each repetition, but the first square of the next repetition(e) is two squares away from the previous first square(a), as shown in figure 18. Hence the algorithm requires two cycles around the cylinder to number all the squares.

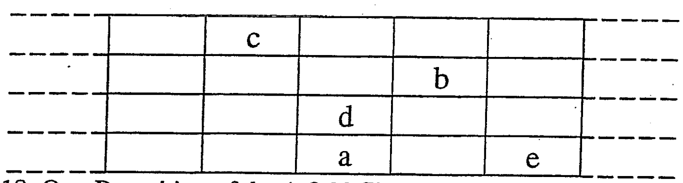

Figure 18. One Repetition of the 4*x*Odd Circuit with the Next Starting Square

At the end of the first cycle, the last repetition completed will begin in the square directly before square one. The next repetition will start the second cycle in the square directly after square one. This second cycle will fill in the squares missed on the previous cycle. The final cycle will end two columns and one row from square one, i.e., a legal knight's move from the start. Now a circuit is complete.

15	3	19	7	23	11	27
10	26	14	2	18	6	22
28	16	4	20	8	24	12
25	13	1	17	5	21	9

Figure 19. Completed Circuit on the $4x7$ Cylinder

The $4x$even cylinder also shows more restrictions than the $3xn$ cylinder. This fact shadows the restrictions found in the board. I developed an algorithm for creating a tour on the $4x$even cylinder, but it is possible to prove that no circuit is contained in there boards.

With the $4x$even cylinder colored in checkerboard fashion, there are an equal number of black and white squares. These two groups can be further partition if the degree of each square is considered. For $n \geq 6$, the squares on the top and bottom of the cylinder have degree 4. Those squares in the two center rows have degree 6. This splits the two groups into four; black squares of degree 4 (A), white of degree 6 (B), black of degree 6 (C), and white of degree 4 (D). These groups are of equal size, say k. If these groups are used as the vertices of a graph, the edges can be defined as the allowed moves between the squares. Elements of A can connect to elements of B, elements of B to elements of C, and elements of C to D. In order for there to be a Hamiltonian circuit of this graph, each vertex must be used once and only once. Since a circuit reaches each vertex, where the circuit "starts" is arbitrary. For this proof it is assumed that there is a circuit beginning in A. Each vertex in the graph can only be the start of an edge once and the end of an edge once. Each of the k vertices in A must then be the start of an edge which ends at a vertex in B. But if the vertices in C and D are to be included in the circuit, then one vertex in B must be the end of an edge from C. This contradicts the requirement that each vertex be the end of only one edge. Hence, no circuits are possible.

Even though there are no circuits on the $4x$even cylinder, the following is an algorithm which gives a tour for the $4x$even cylinder. The coloring in the following algorithm is only used to illustrate the algorithm since, as noted above, the coloring was generally disregarded on these shapes.

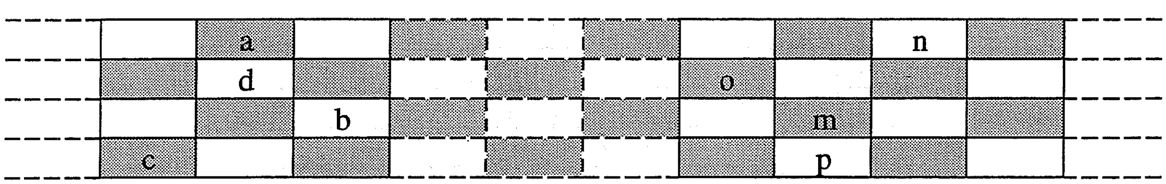

Figure 20. The Two Repetitions of the $4x$Even Tour

The first part of the algorithm will link black squares of lower degree with white squares of higher degree. Start with a black square in the top row. Then move two down and one to the right. Next, go one down and two to the left. Finally, move one to the right and two upward. This last square is now a legal move away from the next black square in the top row. Hence, moving around the cylinder, uses all the low degree black squares and the high degree white squares (see fig. 20). Now it is necessary to fill in the other squares.

The next move goes from the last high degree white square(such as d) to the first high degree black square(such as m). Move one square down and two to the right. Now perform the following moves; one right and two up, one down and two left, and one right and two downward. The last square(p) is now a legal move from the next black square of high degree. In this manner, continue around the cylinder as before until the remaining squares are numbered. Thus the knight's tour is complete.

9	22	1	14	5	18
12	15	4	19	8	23
21	10	13	2	17	6
24	3	16	7	20	11

Figure 21. The Completed Knight's Tour on the 4*x*6 Cylinder

Since there is already a circuit algorithm for the 4*x*odd cylinder, it can be extended to the 4*x*odd torus. Thus, all that remains is to find an algorithm for a circuit on the 4*x*even torus. To do this I went back to the qualities that made the 3*xn* algorithm so nice. Instead of using a method that numbers one square in each row for each repeat, the following algorithm uses two squares in each row. Figure 22 illustrates a single repetition of the algorithm.

		d	g
c	h		
		b	e
a	f		

Figure 22. One Repetition of the 4*x*Even Torus Circuit Algorithm

Since the torus has an even number of columns, the rows will be filled up evenly; there will not be any leftover squares after the last repetition. The other nice quality from the 3*xn* algorithm was also retained in this algorithm. The final square of each repetition is a legal move from the starting square in the next repeat. To reach the starting square of each successive repeat, move one square left and two downward. Thus, when the final repetition is complete, square one is reached by a legal move from the final square and a circuit is complete.

THE M *x* N CASE, M≥5, N≥5

From the paper by Cull and De Curtins [Cu], there is a classification of the chessboards of the form m*x*n, with m and n greater than or equal to five. Cull and De Curtins offer a construction for knight's tours on any mxn board with min(m,n) ≥ 5. They also give a proof for the boards which cannot have a knight's circuit. By the argument given in the 3xn case, no board with m·n odd can have a knight's circuit.

Since it is known that for m·n odd, there is a knight's tour on the board, with no possibility of a circuit, all that remains to be considered are the m·n odd cylinders and tori. An identification of the proper sides in the rectangular representation gives a circuit on the m·n even cylinder and the m·n even torus.

To find a recipe for a circuit algorithm for the odd$x$odd cylinder, I returned to the qualities in the 3xn cylinder algorithm. For each repeat of the algorithm, one square in each row should be used, and the final square of each repetition should be a legal move from the first square of the next repeat. To create such an algorithm for the mxn cylinder, start in the bottom row and alternate up the cylinder, moving two squares up and one to the left, then one to the right and two upward. Repeat these moves until the (m+1)/2 move is reached. That square will be in the top row of the cylinder. This method reaches every other row in the cylinder. Now move down into the row directly below the top row. Next, move down through the rows that were missed on the way up, moving back and forth so that the final square of the repeat is a legal knight's move from the square directly to the right of the first square. Hence, the quality of having each repeat a legal move from the next repeat is preserved. It can also be noted that since the cylinder is filled one square per row per repetition, the algorithm can be used on a cylinder with n even.

		e
f		
	d	
	g	
		c
		h
	b	
	i	
		a

41	50	59	5	14	23	32
60	6	15	24	33	42	51
49	58	4	13	22	31	40
52	61	7	16	25	34	43
39	48	57	3	12	21	30
44	53	62	8	17	26	35
47	56	2	11	20	29	38
54	63	9	18	27	36	45
37	46	55	1	10	19	28

Figure 23. A Cycle for the 9xN Cylinder and a Completed Circuit for 9x7

By working around the cylinder, each row is filled in one square at a time and the last repeat is a legal move from square one. Thus, a circuit algorithm is complete.

To find a circuit on the mxn torus, with m·n odd, a simpler algorithm recipe is available. This recipe takes advantage of the increased mobility on the torus. Starting on the bottom row, begin moving upward on the torus, touching every other row. Alternate between moving two squares up and one to the right, and two up and one to the left. When the (m+1)/2 square is reached, in the top row, continue to move in the same direction, wrapping around the back of the torus. This will number the rows that were missed on the first time around the torus. After the second cycle around the torus, it is possible to come around to the square directly to the right of the starting square of the previous round. This algorithm will also use one square in each row for each repeat and each repetition can be reached by a legal move from the previous one.

	d	
g		
c		
	f	
	b	
e		
a		

25	32	4	11	18
35	7	14	21	28
31	3	10	17	24
27	34	6	13	20
23	30	2	9	16
33	5	12	19	26
29	1	8	15	22

Figure 24. The Repetition for the 7xOdd Torus and the Completed 7x5 Circuit on the Torus

Hence, a circuit algorithm is created. As with the previous algorithms, the algorithms created on the torus can be used when n is either even or odd.

SUMMARY

It is now possible to look back and see what each mapping of the board allows. From the above constructions, it can be seen that any size board on the torus has a knight's circuit. When the torus is "cut" to create a cylinder, that cut adds restrictions to the movements allowed. Consequently, the cylinder has circuits on all boards, except the 4xn, 2xn, and 1xn cases. It can be noted that the 4xn case of the cylinder does have circuits on the 4xodd and tours on the 4xeven. The 2xn case has slightly heavier restrictions. There are no tours on the 2xeven cylinders, but it is possible to find a circuit for the 2xodd.

The next "cut" takes the cylinder to the planar chessboard. Here, there are many more restrictions placed on the boards. For the 1xn and 2xn cases, there are no tours. Also, tours are limited on the 3xn and 4xn boards. In addition, circuits are impossible if one or more of the following conditions hold [Sc]:

1) $m \cdot n$ is odd;
2) m=1,2,4; or,
3) m=3 and n=4,6, or 8.

The chessboard has now been completely classified as to which ones have circuits or not. There still remains to finish doing the same for the tours on the chessboard. Only one case of the board is not classified, the 4xn chessboard. This would be a good topic for another paper.

The deformation of the chessboard opens a wide field of questions. There are many other shapes onto which the board can be mapped. The two shapes discussed here retain the Euclidean geometry of the planar board. There is a possibility of mapping the board onto a surface with a non-Euclidean geometry, such as the two-holed torus. The sphere also presents an interesting framework for further research. The two-holed torus and the sphere do yield some new questions how direction and the legal knight's move should be defined. It also becomes difficult to break these shapes down into cases of like size, as with those in this paper.

Another direction in which this broad question of the knight's tour could head is that of the starting square. Given a board which contains a tour, but is proven to have no circuit, is it possible to start at any square and find a tour of the board. The answer is obviously "no" for those boards where $m \cdot n$ is odd, by the argument given earlier in this paper. There are still cases such as the $4x3$ case which has several tours, but no circuit. There is also the question of how many distinct tours and circuits exist for each board.

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