5 Fully Inhomogeneous Problems

We seek to reduce the problems at hand to problems we already know how to solve. In the process you will meet a couple of new strategies useful in solving initial boundary value problems. As we go along, you should appreciate that we are motivated by the fact that separation of variables is more likely to succeed the more linear homogeneous equations there are in the problem. Also various strategies will be illustrated for one type of PDE or another but you should be aware that a technique applied to one type may work on another type and be prepared to try it.

5.1 Use of Steady-State Solutions

Find the temperature \( u(x,t) \) in a laterally insulated rod of length 1 meter whose left end is kept at temperature \( 10^\circ C \) and whose right end is kept at \( 25^\circ C \). The initial temperature in the rod is \( f(x) \^\circ C \). Heat sources/sinks are given by a function \( F(x) \) independent of time. The IBVP is

\[
\begin{align*}
\mathrm{(IBVP)} \quad \left\{ \begin{array}{ll}
    u_t = \alpha^2 u_{xx} + F(x) & \text{for } 0 < x < 1, \ t > 0, \\
    u(0,t) = 10, \ u(1,t) = 25 & \text{for } t > 0, \\
    u(x,0) = f(x) & \text{for } t \geq 0.
\end{array} \right.
\end{align*}
\]

On physical grounds we anticipate that \( \lim_{t \to \infty} u(x,t) = s(x) \), a steady-state temperature distribution in the rod. (Why?) The steady-state should “remember” HE and BC but will not “remember” the initial condition of the rod. Mathematically speaking a steady-state solution will satisfy HE and BC:

\[
\begin{align*}
\mathrm{(SS)} \quad \left\{ \begin{array}{ll}
    s_t = \alpha^2 s_{xx} + F(x) & \text{for } 0 < x < 1, \ t > 0, \\
    s(0) = 10, \ s(1) = 25 & \text{for } t > 0.
\end{array} \right.
\end{align*}
\]

Notice that \( s_t = 0 \) in SS. Of course, SS is a boundary value problem for the ordinary differential equation \( \alpha^2 s''(x) = -F(x) \). You can solve for \( s(x) \) by antidifferentiating twice and then fitting the two boundary conditions. Now that \( s(x) \) is known, subtract corresponding equations in SS from IBVP to see that \( u(x,t) \) satisfies IBVP iff \( v(x,t) = u(x,t) - s(x) \) satisfies

\[
\begin{align*}
\mathrm{(IBVP-1)} \quad \left\{ \begin{array}{ll}
    v_t = \alpha^2 v_{xx} & \text{for } 0 < x < 1, \ t > 0, \\
    v(0,t) = 0, \ v(1,t) = 0 & \text{for } t > 0, \\
    v(x,0) = f(x) - s(x) & \text{for } t \geq 0.
\end{array} \right.
\end{align*}
\]

You know how to solve IBVP-1 by separation of variables and the solution to IBVP is \( u(x,t) = s(x) + v(x,t) \). The solution \( v \) to IBVP-1 is called the transient part of the solution \( u \) to IBVP; it accommodates the initial conditions in the original problem and decays to zero as \( t \to \infty \).
5.2 Other Reduction Strategies

Consider an IBVP for the wave equation in one spatial dimension for a string of length 1.

\[
\begin{cases}
  u_{tt} = c^2 u_{xx} + F(x,t) & \text{for } 0 < x < 1, \, t > 0, \\
  u(0,t) = r(t), \, u(1,t) = s(t) & \text{for } t > 0, \\
  u(x,0) = f(x), \, u_t(x,0) = g(x) & \text{for } t \geq 0.
\end{cases}
\]

Here we assume \(F, f,\) and \(g\) are continuous and that \(r\) and \(s\) are twice differentiable. We will use the linearity of each equation in IBVP to decompose the problem into three subproblems each of which we can solve. Two of the problems we already know how to solve. Solution of the third is the last piece of the puzzle.

**Reduction 1.** We can solve the fully inhomogeneous IBVP if we can solve a related problem in which the given inhomogeneous boundary conditions are replaced by homogeneous boundary conditions.

This reduction can be done in many ways. The strategy is to write the solution \(u\) to IBVP in the form

\[
u(x,t) = \tilde{u}(x,t) + U(x,t),
\]

where \(U(x,t)\) is a known function that satisfies BC in IBVP because then \(\tilde{u}(x,t)\) will satisfy a similar IBVP with homogeneous BCs. There are many ways to choose a function \(U(x,t)\) that matches the boundary conditions; that is, \(U(0,t) = r(t)\) and \(U(1,t) = s(t)\). One convenient choice is the linear function of \(x\) that interpolates (matches) the data \(r(t)\) and \(s(t)\) at the ends of the string:

\[
U(x,t) = (1-x)r(t) + xs(t).
\]

A simple calculation confirms that \(u(x,t) = \tilde{u}(x,t) + U(x,t)\) solves IBVP iff \(\tilde{u}(x,t)\) solves

\[
\begin{cases}
  \tilde{u}_{tt} = c^2 \tilde{u}_{xx} + \tilde{F}(x,t) & \text{for } 0 < x < 1, \, t > 0, \\
  \tilde{u}(0,t) = 0, \, \tilde{u}(1,t) = 0 & \text{for } t > 0, \\
  \tilde{u}(x,0) = \tilde{f}(x), \, \tilde{u}_t(x,0) = \tilde{g}(x) & \text{for } t \geq 0,
\end{cases}
\]

where

\[
\begin{align*}
\tilde{F}(x,t) &= F(x,t) - U_{tt}(x,t) + c^2 U_{xx}(x,t), \\
\tilde{f}(x) &= f(x) - U(x,0), \\
\tilde{g}(x) &= g(x) - U_t(x,0).
\end{align*}
\]

If we can solve IBP for \(\tilde{u}\), then \(u(x,t) = \tilde{u}(x,t) + U(x,t)\) solves IBVP.
Reduction 2. Solution of IBVP~:

We take advantage of the fact that all equations in IBVP~ are \(\text{linear in } \tilde{u}\) to split this problem into two subproblems:

\[
\begin{align*}
\text{(IBVP~1)} & \quad \begin{cases} v_{tt} = c^2 v_{xx} & \text{for } 0 < x < 1, \ t > 0, \ (\text{WE}) \\
v (0, t) = 0, \ v (1, t) = 0 & \text{for } t > 0, \ (\text{BC}) \\
v (x, 0) = \tilde{f}(x), \ v_t (x, 0) = \tilde{g}(x) & \text{for } t \geq 0. \ (\text{IC})
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{(IBVP~2)} & \quad \begin{cases} w_{tt} = c^2 w_{xx} + \tilde{F}(x,t) & \text{for } 0 < x < 1, \ t > 0, \ (\text{WE}) \\
w (0, t) = 0, \ w (1, t) = 0 & \text{for } t > 0, \ (\text{BC}) \\
w (x, 0) = 0, \ w_t (x, 0) = 0 & \text{for } t \geq 0. \ (\text{IC})
\end{cases}
\end{align*}
\]

Add corresponding equations in these two auxiliary problems to see that \(\tilde{u}(x,t) = v(x,t) + w(x,t)\) solves IBVP~ if \(v\) and \(w\) solve their respective problems. We have already seen how to solve IBVP~1 by separation of variables.

Final Step. We can solve IBVP~2 by a separation of variables approach:

Motivated by the form of the separation-of-variables solution to the \emph{homogeneous} wave equation and homogeneous boundary conditions in IBVP~1 and by the method of variation of parameters in ODEs, we are led to try for a solution to IBVP~2 of the form

\[
w (x,t) = \sum_{n=1}^{\infty} w_n (t) \sin \frac{n\pi x}{1} = \sum_{n=1}^{\infty} w_n (t) \sin n\pi x
\]

where the functions (variable parameters), \(w_n (t)\), must be chosen so that the series solves IBVP~2. Let’s see if we can do it.

What is the parallel develop that you would follow to solve a fully inhomogeneous IBVP for the heat equation in one spatial dimension? When would you follow the parallel approach instead of the steady state approach?

A combination of ideas from Sec. 5.1 and Sec. 5.2 can be useful.